**Definition 2.1.** A collection $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is said to be an $\omega_1$-tree if all of the following hold:

- $\mathcal{T}$ has height $\omega_1$;
- $\mathcal{T}$ is downward closed;
- for all $\beta < \omega_1$, $\mathcal{T}_\beta$ (that is, $\mathcal{T} \cap {}^\beta\omega$) is countable.

Recall that an $\omega_1$-tree is said to be *Aronszajn* if it does not admit a cofinal branch.

In the previous lecture, we proved that

$$\mathcal{T}(\rho_2) := \{\rho_{2\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$$

is an $\omega_1$-tree. Let us point out that the levels (but to the bottom one) of $\mathcal{T}(\rho_2)$ are infinite.

**Lemma 2.2.** *If* $0 < \beta \leq \delta < \omega_1$, *then* $\rho_{2\delta} \restriction \beta \neq \rho_{2(\delta+n)} \restriction \beta$ *whenever* $0 < n < \omega$.

*Proof.* Follows immediately from the following. □

**Lemma 2.3.** *For all* $\alpha < \delta < \omega_1$ *and* $n < \omega$, *we have* $\rho_{2(\delta+n)}(\alpha) = \rho_{2\delta}(\alpha) + n$.

*Proof.* By induction on $n < \omega$. The case $n = 0$ is trivial. Now, if $n < \omega$ and $\rho_{2\delta+n}(\alpha) = \rho_{2\delta}(\alpha) + n$, then by $C_{\delta+n+1} = \{\delta + n\}$, we have $\mathrm{Tr}(\alpha, \delta + n + 1)(1) = \min(C_{\delta+n+1} \setminus \alpha) = \delta + n$, and so by Lemma 1.5 of the previous lecture:

- $\mathrm{tr}(\alpha, \delta + n + 1) = \mathrm{tr}(\delta + n, \delta + n + 1)^\frown \mathrm{tr}(\alpha, \delta + n)$, and
- $\rho_2(\alpha, \delta + n + 1) = 1 + \rho_2(\alpha, \delta + n)$.

By the hypothesis, then, $\rho_{2\delta+n+1}(\alpha) = \rho_{2\delta+n}(\alpha)+1 = \rho_{2\delta}(\alpha)+n+1$. □

We now introduce another characteristic function of the walk.

**Definition 2.4.** Define $\lambda : [\omega_1]^2 \to \omega_1$ by stipulating:

$$\lambda(\beta, \delta) := \max\{\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) \mid i < \rho_2(\beta, \delta)\}.$$

**Lemma 2.5.** *If* $0 < \beta < \delta < \omega_1$, *then* $\lambda(\beta, \delta) < \beta$.

*Proof.* For all $i < \rho_2(\beta, \delta)$, $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta$ is a finite subset of $\beta$. So $\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) < \beta$. Consequently, $\lambda(\beta, \delta)$ is the maximum over a finite subset of $\beta$, and hence $< \beta$. □

**Lemma 2.6.** *If* $\lambda(\beta, \delta) < \alpha < \beta < \delta < \omega_1$, *then* $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\beta, \delta)^\frown \mathrm{tr}(\alpha, \beta)$.

*Proof.* We shall show that $\mathrm{Tr}(\alpha, \delta)(n) = \mathrm{Tr}(\beta, \delta)(n)$ for all $n \leq \rho_2(\beta, \delta)$. In particular, $\mathrm{tr}(\alpha, \delta)(\rho_2(\beta, \delta)) = \beta$, and then the conclusion will follow from Lemma 1.5 of the previous lecture.

► Clearly, we have $\mathrm{Tr}(\alpha, \delta)(0) = \delta = \mathrm{Tr}(\beta, \delta)(0)$.

▶ Next, if $n < \rho_2(\beta,\delta)$, and $\mathrm{Tr}(\alpha,\delta)(n) = \mathrm{Tr}(\beta,\delta)(n)$, say, it is equal to $\gamma$, then:

- $\mathrm{Tr}(\alpha,\delta)(n+1) = \min(C_\gamma \setminus \alpha)$;
- $\mathrm{Tr}(\beta,\delta)(n+1) = \min(C_\gamma \setminus \beta)$.

Trivially, $C_\gamma \cap [\sup(C_\gamma \cap \beta), \beta) = \emptyset$. As $\sup(C_\gamma \cap \beta) = \sup(C_{\mathrm{Tr}(\beta,\delta)(n)} \cap \beta) \leq \lambda(\beta,\delta) < \alpha$, we get in particular that $C_\gamma \cap [\alpha,\beta) = \emptyset$. So $\min(C_\gamma \setminus \alpha) = \min(C_\gamma \setminus \beta)$, and hence $\mathrm{Tr}(\alpha,\delta)(n+1) = \mathrm{Tr}(\beta,\delta)(n+1)$. □

Complementary to Lemma 2.3, we have the following.

**Corollary 2.7.** *For every $\beta < \delta < \omega_1$, the set $\{\rho_{2\beta}(\alpha) - \rho_{2\delta}(\alpha) \mid \alpha < \beta\}$ is finite.*

*Proof.* Suppose that $\beta < \delta < \omega_1$ gives a counterexample. Then, for every $k < \omega$, we can pick $\alpha_k < \beta$ such that $|\rho_{2\beta}(\alpha_k) - \rho_{2\delta}(\alpha_k)| > k$. Define $c : [\omega]^2 \to 2$ by letting for all $n < m < \omega$: $c(n,m) := 0$ iff $\alpha_n \leq \alpha_m$. By Ramsey's theorem, there exists an infinite $H \subseteq \omega$ which is homogeneous for $c$. Since there cannot be a strictly decreasing sequence of ordinals, $H$ must be 0-homogeneous. So $\langle \alpha_k \mid k \in H\rangle$ is infinite and non-decreasing. Since $\sup_{k\in H} |\rho_{2\beta}(\alpha_k) - \rho_{2\delta}(\alpha_k)| = \omega$, we may pass to an infinite subset of $H$, and moreover assume that $\langle \alpha_k \mid k \in H\rangle$ is increasing.

Let $\alpha^* := \sup\{\alpha_k \mid k \in H\}$. Consider the positive integer

$$n := |\rho_2(\alpha^*,\delta) - \rho_2(\alpha^*,\beta)|.$$

Let

$$\lambda := \begin{cases} \lambda(\alpha^*,\delta), & \text{if } \alpha^* = \beta, \\ \max\{\lambda(\alpha^*,\delta), \lambda(\alpha^*,\beta)\}, & \text{if } \alpha^* < \beta. \end{cases}$$

As $\lambda < \alpha^*$, let us pick a large enough $k \in H$ such that $k > n$ and $\alpha_k > \lambda$.

By $\lambda(\alpha^*,\delta) \leq \lambda < \alpha_k < \alpha^* < \delta$ and by Lemma 2.6, we have:

$$\mathrm{tr}(\alpha_k,\delta) = \mathrm{tr}(\alpha^*,\delta)^\frown \mathrm{tr}(\alpha_k,\alpha^*).$$

Next, we analyze $\mathrm{tr}(\alpha_k,\beta)$. There are two cases to consider:

▶ If $\alpha^* = \beta$, then $\mathrm{tr}(\alpha^*,\beta) = \emptyset$, and trivially

$$\mathrm{tr}(\alpha_k,\beta) = \mathrm{tr}(\alpha^*,\beta)^\frown \mathrm{tr}(\alpha_k,\alpha^*).$$

▶ If $\alpha^* < \beta$, then by $\lambda(\alpha^*,\beta) \leq \lambda < \alpha_k < \alpha^* < \beta$ and by Lemma 2.6, we have:

$$\mathrm{tr}(\alpha_k,\beta) = \mathrm{tr}(\alpha^*,\beta)^\frown \mathrm{tr}(\alpha_k,\alpha^*).$$

Altogether, $|\rho_{2\delta}(\alpha_k) - \rho_{2\beta}(\alpha_k)| = |\rho_2(\alpha^*,\delta) - \rho_2(\alpha^*,\beta)| = n < k$, contradicting the very choice of $\alpha_k$. □

As seen in the first lecture, to establish that $\mathcal{T}(\rho_2)$ is Aronszajn, it suffices to prove that for every uncountable $X \subseteq \omega_1$ and every $k < \omega$, there exist $\alpha < \beta$ both from $X$ such that $\rho_2(\alpha, \beta) \geq k$. This will be done in Lemma 2.9 below. The proof of this lemma assumes familiarity with *club* and *stationary sets*, which we now introduce in brief. Recall that $\beta < \omega_1$ is said to be a *closure point* of a function $f : \omega_1 \to \omega_1$, if $f(\alpha) < \beta$ for all $\alpha < \beta$. That is, $f[\beta] \subseteq \beta$. Now, a set $C \subseteq \omega_1$ is said to be a *club* if it is the set of closure points of some function $f : \omega_1 \to \omega_1$. That is, $C = \{\beta < \omega_1 \mid f[\beta] \subseteq \beta\}$. For instance, the set of all limit ordinals below $\omega_1$ is a club, as witnessed by the map $\alpha \mapsto \alpha + 1$. A subset $S \subseteq \omega_1$ is said to be *stationary* if it meets any club, that is, if for every $f : \omega_1 \to \omega_1$, there exists some $\beta \in S$ with $f[\beta] \subseteq \beta$. It is easy to see that the the intersection of two clubs is a club, and that the intersection of countably many clubs contains a club. In particular, any club is stationary, and the intersection of a club and a stationary set is again stationary. We point out that there is a notion of *c.u.b.* which is closely related to that of a *club*. A subset $C \subseteq \omega_1$ is said to be *closed* if for any increasing $\langle \alpha_n \mid n < \omega \rangle$ of elements from $C$, the limit $\sup_{n<\omega} \alpha_n$ is in $C$. A set $C \subseteq \omega_1$ is *unbounded* if for all $\alpha < \omega_1$, there exists $\beta \in C$ with $\alpha < \beta$. Any club set is c.u.b., and any c.u.b. contains a club. In particular, the collection of all clubs and the collection of all c.u.b.'s generate the same filter.

**Exercise 2.8.** *Show that clubs and stationary sets are necessarily uncountable.*

An instrumental generalization of the Pigeonhole Principle is *Fodor's lemma* that asserts that for every stationary $S \subseteq \omega_1$ and every $g : S \to \omega_1$ that satisfies $g(\alpha) < \alpha$ for all nonzero $\alpha \in S$, there exists some stationary $T \subseteq S$ on which $g \restriction T$ is constant.

*Notation.* For sets of ordinals $a, b$, write $a < b$ whenever $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$.

**Lemma 2.9.** *Suppose that $X$ is an uncountable family of pairwise disjoint finite subsets of $\omega_1$. For all $k < \omega$, there exists an uncountable $Y \subseteq X$ such that* $\min(\rho_2[a \times b]) \geq k$ *for all $a < b$ both from $Y$.*

*Proof.* The statement is trivially valid for $k = 0$. Next, suppose that the statement is valid for $k$, and let us prove it for $k + 1$. Since $X$ is uncountable and consists of pairwise disjoint sets, for each $\beta < \omega_1$, we may pick some $x_\beta \in X$ with $\min(x_\beta) > \beta$. Define a function $f : \omega_1 \to \omega_1$ by stipulating $f(\beta) := \max(x_\beta)$, and consider the club $C := \{\beta < \omega_1 \mid f[\beta] \subseteq \beta\}$, so that for all $\beta < \beta'$ both from $C$, we have $\{\beta\} < x_\beta < \{\beta'\} < x_{\beta'}$.

Next, for all nonzero limit ordinal $\beta \in C$, put $\lambda_\beta := \max\{\lambda(\beta, \alpha) \mid \alpha \in x_\beta\}$. By Lemma 2.5, $\lambda_\beta < \beta$. By Fodor's lemma, let us pick a stationary subset $S \subseteq C$ such that $\{\lambda_\beta \mid \beta \in S\}$ is a singleton, say $\{\lambda\}$, and put $X' := \{\{\beta\} \cup x_\beta \mid \beta \in S\}$. By $S \subseteq C$, we know that $X'$ consists of pairwise disjoint finite sets, so we may appeal to the induction hypothesis and find an uncountable $Y' \subseteq X'$ such that $\min(\rho_2[a \times b]) \geq \kappa$ whenever $a < b$ are both from $Y'$. Let $Y := \{y \setminus \{\min(y)\} \mid y \in Y'\}$, so that $Y$ is an uncountable subset of $X$.

Now, let $a < b$ be arbitrary elements of $Y$. Pick arbitrary $\alpha \in a$ and $\alpha' \in b$, and let us show $\rho_2(\alpha, \alpha') \geq k + 1$. Fix $\beta < \beta'$ such that $a = x_\beta$ and $b = x_{\beta'}$. By

- $\beta, \beta' \in S$,
- $\alpha \in x_\beta$,
- $\beta' \in C$, and
- $\alpha' \in x_{\beta'}$,

we have

$$\lambda(\beta', \alpha') \leq \lambda_{\beta'} = \lambda = \lambda_\beta < \beta < \alpha < \beta' < \alpha'.$$

So, by Lemma 2.6, we get that $\mathrm{tr}(\alpha, \alpha') = \mathrm{tr}(\beta', \alpha')^\frown \mathrm{tr}(\alpha, \beta')$, and hence $\rho_2(\alpha, \alpha') \geq 1 + \rho_2(\alpha, \beta')$. Finally, as $\{\beta\} \cup a$ and $\{\beta'\} \cup b$ are in $Y'$, and $\alpha \in a$, we have $\rho_2(\alpha, \beta') \geq k$. Altogether, $\mathrm{tr}(\alpha, \alpha') \geq k+1$. □

**Corollary 2.10.** *$\mathcal{T}(\rho_2)$ is an Aronszajn tree.* □

Another interesting corollary is that there exists no $b : \omega_1 \to \omega$ such that $\{\beta < \delta \mid \rho_{2\delta}(\beta) \neq b(\beta)\}$ is finite for all $\delta < \omega_1$.

**Corollary 2.11.** *For every $b : \omega_1 \to \omega$, there exists some $\delta < \omega_1$ such that*

$$\sup\{|\rho_{2\delta}(\beta) - b(\beta)| \mid \beta < \delta\} = \infty.$$

*Proof.* Suppose not. Then there exists $b : \omega_1 \to \omega$ such that for all $\delta < \omega_1$, the following is an integer:

$$n_\delta := \sup\{|\rho_{2\delta}(\beta) - b(\beta)| \mid \beta < \delta\}.$$

Fix an uncountable $A \subseteq \omega_1$ on which $\{n_\delta \mid \delta \in A\}$ is a singleton, say, $\{n\}$. Then, fix an uncountable $B \subseteq A$ on which $\{b(\beta) \mid \beta \in B\}$ is a singleton, say, $\{m\}$. By Lemma 2.9, there exist $\beta < \delta$ both from $B$ such that $\rho_2(\beta, \delta) \geq n + m + 1$. By $\delta \in B \subseteq A$ and $\beta < \delta$, we have $|\rho_{2\delta}(\beta) - b(\beta)| \leq n$. By $\beta \in B$, we have $b(\beta) = m$, so that $\rho_2(\beta, \delta) \leq n + m$. This is a contradiction. □

**Definition 2.12.** An $\omega_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is said to be $\mathbb{R}$-embeddable. if there exists an order-preserving mapping $f : \mathcal{T} \to \mathbb{R}$ from the tree to the real line.[1]

**Lemma 2.13.** *If an $\omega_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is $\mathbb{R}$-embeddable, then $\mathcal{T}$ is Aronszajn.*

*Proof.* Suppose that $f : \mathcal{T} \to \mathbb{R}$ is order-preserving. Suppose towards a contradiction that $b : \omega_1 \to \omega$ is a cofinal branch through $\mathcal{T}$. Since $f$ is order-preserving, for all $\alpha < \omega_1$, we have $f(b \restriction \alpha) < f(b \restriction \alpha + 1)$, so let us pick a rational number $q_\alpha$ such that

$$f(b \restriction \alpha) < q_\alpha < f(b \restriction \alpha + 1).$$

Notice that if $\alpha < \beta < \omega_1$, then

$$f(b \restriction \alpha) < q_\alpha < f(b \restriction \alpha + 1) \leq f(b \restriction \beta) < q_\beta.$$

So, the mapping $\alpha \mapsto q_\alpha$ is an injection from $\omega_1$ to $\mathbb{Q}$, contradicting the fact that $|\omega_1| > |\mathbb{Q}|$. □

A subset $A$ of ${}^{<\omega_1}\omega$ is said to be an *antichain* if for all two distinct $\sigma, \sigma'$ from $A$, $\sigma \nsubseteq \sigma'$ and $\sigma' \nsubseteq \sigma$.

**Definition 2.14.** An $\omega_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is said to be *special* if $\mathcal{T}$ may be covered by countably many antichains.

**Exercise 2.15.** *Show that any special $\omega_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is $\mathbb{R}$-embeddable, and hence Aronszajn.*

**Exercise 2.16.** *Show that $\mathcal{T}$ is special iff there exists a countable set $\Upsilon$, and a function $c : \mathcal{T} \to \Upsilon$ such that the preimage of any singleton is an antichain.*

We would like to show that $\mathcal{T}(\rho_2)$ is special. For this, we shall need the following.

**Lemma 2.17.** *For every $\beta < \delta < \omega_1$, the set $\{\alpha < \beta \mid \rho_{2\beta}(\alpha) = \rho_{2\delta}(\alpha)\}$ is closed.*

*Proof.* Suppose that $\alpha^* < \beta$ be an accumulation point of $A := \{\alpha < \beta \mid \rho_{2\beta}(\alpha) = \rho_{2\delta}(\alpha)\}$. That is, $\sup(A \cap \alpha^*) = \alpha^*$. We must show that $\alpha^* \in A$.

By Lemma 2.5, $\lambda(\alpha^*, \delta) < \alpha^*$ and $\lambda(\alpha^*, \beta) < \alpha^*$. Thus, we may fix a large enough $\alpha \in A$ which is greater than $\lambda(\alpha^*, \beta)$ and $\lambda(\alpha^*, \delta)$. It then follows from Lemma 2.6 that:

- $\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\alpha^*, \beta)^\frown \mathrm{tr}(\alpha, \alpha^*)$;

[1] Note: $f$ is not required to be one-to-one. It is only required to be one-to-one over chains.

- $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha, \alpha^*)$.

So $\rho_2(\alpha^*, \beta) = \rho_2(\alpha, \beta) - \rho_2(\alpha, \alpha^*) = \rho_2(\alpha, \delta) - \rho_2(\alpha, \alpha^*) = \rho_2(\alpha^*, \delta)$.
$\square$