Last week, we proved that $\mathcal{T}(\rho_2)$ is an Aronszajn tree. In addition, we proved the following.

Lemma 3.1. *For every $\beta < \delta < \omega_1$, the set $\{\alpha < \beta \mid \rho_{2\beta}(\alpha) = \rho_{2\delta}(\alpha)\}$ is closed.* □

Today, we shall prove that $\mathcal{T}(\rho_2)$ is special.[1] As will only become clear later, this will follow from the following trivial observation.

Lemma 3.2. *For all $n < \omega$, $\{\rho_{2(\delta+n)} \mid \delta < \omega_1$ limit nonzero$\}$ is an antichain.*

Proof. Let $\beta < \delta < \omega_1$ be arbitrary nonzero limit ordinals. By a lemma from last week, $\lambda(\beta, \delta) < \beta$. Since β is a limit ordinal, we may fix α with $\lambda(\beta, \delta) < \alpha < \beta$. By another lemma from last week, then, $\operatorname{tr}(\alpha, \delta) = \operatorname{tr}(\beta, \delta)^\frown \operatorname{tr}(\alpha, \beta)$, and hence $\rho_{2\delta}(\alpha) > \rho_{2\beta}(\alpha)$. Consequently, $\rho_{2(\delta+n)}(\alpha) = \rho_{2\delta}(\alpha) + n > \rho_{2\beta}(\alpha) + n = \rho_{2(\beta+n)}(\alpha)$ for all $n < \omega$. □

In his 2013 dissertation, Peng introduced a mapping $\pi : {}^{<\omega_1}\omega \to {}^{<\omega_1}\mathbb{Z}$ that exploits Lemma 3.1. Given $f : \beta \to \omega$, let $\pi(f) : \beta \to \mathbb{Z}$ be the unique function that satisfies for all $\alpha < \beta$:

$$\pi(f)(\alpha) = f(\alpha) - f(\sup(\alpha)).$$

That is, we let $\pi(f)(\alpha) := 0$ for all limit $\alpha < \beta$, and let $\pi(f)(\alpha) := f(\alpha) - f(\alpha - 1)$ for all successor $\alpha < \beta$.

Write $\mathcal{T}_\pi(\rho_2) := \{\pi(f) \mid f \in \mathcal{T}(\rho_2)\}$.

Lemma 3.3 (Peng)**.** *$(\mathcal{T}(\rho_2), \subseteq)$ and $(\mathcal{T}_\pi(\rho_2), \subseteq)$ are order-isomorphic. In particular, $\mathcal{T}(\rho_2)$ is special iff $\mathcal{T}_\pi(\rho_2)$ is special.*

Proof. For all $\alpha < \beta < \omega_1$, and $f : \beta \to \omega$, the value of $\pi(f)(\alpha)$ depends only on $f \restriction (\alpha + 1)$. In particular, $f \subseteq g$ entails $\pi(f) \subseteq \pi(g)$ for any $f, g \in \mathcal{T}(\rho_2)$.

Next, suppose that we are given $f, g \in \mathcal{T}(\rho_2)$ with $f \nsubseteq g$. We shall show that $\pi(f) \nsubseteq \pi(g)$. Pick $\beta \le \delta < \omega_1$ and $\beta' \le \delta' < \omega_1$ such that $f = \rho_{2\delta} \restriction \beta$ and $g = \rho_{2\delta'} \restriction \beta'$.

▶ If $\beta > \beta'$, then trivially $\pi(f) \nsubseteq \pi(g)$.

▶ Suppose that $\beta \le \beta'$, and let $\alpha < \beta$ be the least such that $f(\alpha) \neq g(\alpha)$. By Lemma 3.1, α is a successor ordinal, so that $\pi(f)(\alpha) = f(\alpha) - f(\alpha - 1)$ and $\pi(g)(\alpha) = g(\alpha) - g(\alpha - 1)$. Now, by $f(\alpha) \neq g(\alpha)$ and $f(\alpha - 1) = g(\alpha - 1)$, we conclude that $\pi(f)(\alpha) \neq \pi(g)(\alpha)$, as sought. □

[1] Recall that $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is special iff there exists a countable set Υ and a function $c : \mathcal{T} \to \Upsilon$ such that the preimage of any singleton is an antichain.

Definition 3.4. A family $\mathcal{T}$ of functions is called *coherent* if for all $f, g \in \mathcal{T}$, the set $\{i \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \mid f(i) \neq g(i)\}$ is finite.

Lemma 3.5 (Peng)**.** *$\mathcal{T}_\pi(\rho_2)$ is coherent.*

Proof. Suppose not. Pick ordinals $\beta \leq \delta < \omega_1$ and $\beta' \leq \delta' < \omega_1$, such that the following set is infinite:

$$\{\alpha < \min\{\beta, \beta'\} \mid \pi(\rho_{2\delta} \restriction \beta)(\alpha) \neq \pi(\rho_{2\delta'} \restriction \beta')(\alpha)\}.$$

In particular,

$$A := \{\alpha < \min\{\delta, \delta'\} \mid \pi(\rho_{2\delta})(\alpha) \neq \pi(\rho_{2\delta'})(\alpha)\}$$

is infinite. Recalling the definition of π, A consists of successor ordinals. Let α^* be the least ordinal such that $A \cap \alpha^*$ is infinite. Then α^* is a limit ordinal. Pick $\alpha \in A \cap \alpha^*$ with $(\alpha - 1) > \max\{\lambda(\alpha^*, \delta), \lambda(\alpha^*, \delta')\}$. By Lemma 2.6 of the previous lecture, then:

$$\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha, \alpha^*), \quad \mathrm{tr}(\alpha - 1, \delta) = \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha - 1, \alpha^*),$$
$$\mathrm{tr}(\alpha, \delta') = \mathrm{tr}(\alpha^*, \delta')^\frown \mathrm{tr}(\alpha, \alpha^*), \quad \mathrm{tr}(\alpha - 1, \delta') = \mathrm{tr}(\alpha^*, \delta')^\frown \mathrm{tr}(\alpha - 1, \alpha^*).$$

Consequently:

$$\begin{aligned}\pi(\rho_{2\delta})(\alpha) = \rho_{2\delta}(\alpha) - \rho_{2\delta}(\alpha - 1) = \\ \rho_{2\delta}(\alpha^*) + \rho_{2\alpha^*}(\alpha) - (\rho_{2\delta}(\alpha^*) + \rho_{2\alpha^*}(\alpha - 1)) = \\ \rho_{2\delta'}(\alpha^*) + \rho_{2\alpha^*}(\alpha) - (\rho_{2\delta'}(\alpha^*) + \rho_{2\alpha^*}(\alpha - 1)) = \\ \rho_{2\delta'}(\alpha) - \rho_{2\delta'}(\alpha - 1) = \pi(\rho_{2\delta'})(\alpha),\end{aligned}$$

contradicting the fact that $\alpha \in A$. □

Next, we establish a sufficient condition for coherent trees to be special.

Lemma 3.6 (Peng)**.** *Suppose that $\mathcal{T} \subseteq {}^{<\omega_1}N$ is a coherent $\aleph_1$-tree, for some countable set N.*

Then $\mathcal{T}$ is special iff there exists a c.u.b. $E \subseteq \omega_1$ and a choice $b \in \prod_{\delta \in E} \mathcal{T}_\delta$ whose range is the union of countably many antichains.

Proof. The forward implication is trivial, so we focus on the converse.

Write $V_0 := \emptyset$, and $V_{n+1} := \mathcal{P}(V_n)$ for all $n < \omega$. Then $|V_{n+1}| = 2^{|V_n|}$ is finite, and $V_\omega := \bigcup_{n<\omega} V_n$ is countable. In addition, for any $n < \omega$ and any function $f : n \to V_\omega$, we have $f \in V_\omega$.[2]

We shall want to define a function $c : \mathcal{T} \to V_\omega$ such that the preimage of any singleton is an antichain.

Fix a c.u.b. $E \subseteq \omega_1$, a choice $b \in \prod_{\delta \in E} \mathcal{T}_\delta$, and $d : \mathrm{Im}(b) \to \omega$ such that the preimage of any singleton is an antichain. We may clearly

[2] In fact, $(V_\omega, \in)$ is a model of ZFC minus the axiom of infinity.

assume that $0 \in E$. For every $\delta < \omega_1$, fix an injection $\psi_\delta : \delta + 1 \to \omega$, and write $\bar\delta := \sup(E \cap \delta)$. Note that $\bar\delta \le \delta$, and $\bar\delta \in E$, since E is closed.

Next, for all $\delta \in E$ and $x \in \mathcal{T}_\delta$, write b_δ for $b(\delta)$, and consider the finite set $\Delta_x := \{\alpha < \delta \mid x(\alpha) \neq b_\delta(\alpha)\}$. For all $i < |\Delta_x|$, let $\Delta_x(i)$ denote the i^{th} element of Δ_x.

We now define $c \restriction \mathcal{T}_\delta$ by recursion on $\delta < \omega_1$. Given $x \in \mathcal{T}_\delta$, denote $\bar x := x \restriction \bar\delta$, and let $c(x)$ be the function from $2|\Delta_{\bar x}| + 2$ to V_ω defined by:

$$c(x)(i) := \begin{cases} b_{\bar\delta}(\Delta_{\bar x}(i)), & \text{if } i < |\Delta_{\bar x}|; \\ c(\bar x \restriction \Delta_{\bar x}(j)), & \text{if } i = |\Delta_{\bar x}| + j < 2|\Delta_{\bar x}| \\ d(b_{\bar\delta}), & \text{if } i = 2|\Delta_{\bar x}|; \\ \psi_{\min(E\setminus\delta)}(\delta), & \text{if } i = 2|\Delta_{\bar x}| + 1. \end{cases}$$

Towards a contradiction, suppose that there exist $x \subset y$ in $\mathcal{T}$ such that $c(x) = c(y)$. Pick such a pair (x, y) with $(\mathrm{dom}(x), \mathrm{dom}(y))$ of lexicographically least possible value, say, (δ, δ'). In particular, $\delta \le \delta'$. As x is a proper subsequence of y, we have $\delta < \delta'$.

Write $n := |\Delta_{\bar x}|$, so that $c(x) = c(y)$ is a function from $2n+2$ to V_ω.

Claim 3.6.1. *$\bar\delta < \bar\delta'$;*

Proof. Otherwise, $\bar\delta = \bar\delta'$, That is, $\sup(E \cap \delta) = \sup(E \cap \delta')$. But, then $\min(E \setminus \delta) = \min(E \setminus \delta')$ and so by $\delta < \delta'$, we should have had $\psi_{\min(E\setminus\delta)}(\delta) \neq \psi_{\min(E\setminus\delta')}(\delta')$, contradicting the fact that $c(x)(2n+1) = c(y)(2n+1)$. □

Claim 3.6.2. *There exists some $j < n$ such that $\bar x \restriction \Delta_{\bar x}(j) \neq \bar y \restriction \Delta_{\bar y}(j)$.*[3]

Proof. We have $\bar\delta < \bar\delta'$, so that $b_{\bar\delta} \neq b_{\bar\delta'}$. But $d(b_{\bar\delta}) = c(x)(2n) = c(y)(2n) = d(b_{\bar\delta'})$, and hence $b_{\bar\delta}$ must be incompatible with $b_{\bar\delta'}$. Let $\gamma < \bar\delta$ be the least such that $b_{\bar\delta}(\gamma) \neq b_{\bar\delta'}(\gamma)$. By $x \subseteq y$, we have $\bar x \restriction (\gamma+1) = \bar y \restriction (\gamma+1)$, and hence for all $\alpha < \gamma$:

$$\bar x(\alpha) = b_{\bar\delta}(\alpha) \text{ iff } \bar x(\alpha) = b_{\bar\delta'}(\alpha) \text{ iff } \bar y(\alpha) = b_{\bar\delta'}(\alpha).$$

So $\Delta_{\bar x} \cap \gamma = \Delta_{\bar y} \cap \gamma$. We claim that $j := |\Delta_{\bar x} \cap \gamma|$ is as sought.

▶ If $\bar x(\gamma) = b_{\bar\delta}(\gamma)$, then $\bar x(\gamma) \neq b_{\bar\delta'}(\gamma)$, and by $x \restriction (\gamma+1) = y \restriction (\gamma+1)$, then, $\bar y(\gamma) \neq b_{\bar\delta'}(\gamma)$. So, in this case, $\gamma \in \Delta_{\bar y} \setminus \Delta_{\bar x}$ and $\Delta_{\bar y}(j) = \gamma < \Delta_{\bar x}(j)$. Consequently, $\bar y \restriction \Delta_{\bar y}(j) \subset \bar x \restriction \Delta_{\bar x}(j)$.

▶ If $\bar x(\gamma) \neq b_{\bar\delta}(\gamma)$, then $\gamma \in \Delta_{\bar x}$, so that $\Delta_{\bar x}(j) = \gamma$. Now, if $\Delta_{\bar y}(j)$ were to equal γ, as well, then $b_{\bar\delta}(\gamma) = c(x)(j) = c(y)(j) = b_{\bar\delta'}(\gamma)$,

[3] In particular, $n > 0$.

contradicting the choice of γ. So, in this case, $\bar{x} \restriction \Delta_{\bar{x}}(j) \subset \bar{y} \restriction \Delta_{\bar{y}}(j)$. $\square$

Let $j < n$ be given by the preceding. Then $\bar{x} \restriction \Delta_{\bar{x}}(j)$ and $\bar{y} \restriction \Delta_{\bar{y}}(j)$ are two distinct compatible elements, and

$$c(\bar{x} \restriction \Delta_{\bar{x}}(j)) = c(x)(n+j) = c(y)(n+j) = c(\bar{y} \restriction \Delta_{\bar{y}}(j)).$$

But then, the pair $(\Delta_{\bar{x}}(j), \Delta_{\bar{y}}(j)) \in \bar{\delta} \times \bar{\delta}'$ contradicts the minimality of the pair (δ, δ'). $\square$

Corollary 3.7 (Peng). *$\mathcal{T}(\rho_2)$ is special.*

Proof. By Lemma 3.2, $\{\rho_{2\delta} \mid \delta < \omega_1\}$ is the countable union of antichains. Since π is an isomorphism, so is $\{\pi(\rho_{2\delta}) \mid \delta < \omega_1\}$. By Lemma 3.6, then, $\mathcal{T}_\pi(\rho_2)$ is special. Consequently, $\mathcal{T}(\rho_2)$ is special. $\square$

Observation 3.8. *If there exists a nonspecial Aronszajn tree, then there exists an Aronszajn tree that is not isomorphic to a coherent one.*

Proof. Suppose that there exists a nonspecial Aronszajn tree. In particular, we may pick $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ which is Aronszajn and nonspecial. Also, pick $\mathcal{S} \subseteq {}^{<\omega_1}\omega$ which is Aronszajn and special, e.g., $\mathcal{S} = \mathcal{T}(\rho_2)$.

Clearly, $\mathcal{T} \cup \mathcal{S}$ is an ω_1-tree. By the pigeonhole principle, it is moreover Aronszajn.

Now, if $\mathcal{T} \cup \mathcal{S}$ was isomorphic to a coherent tree $\mathcal{C}$, then by Lemma 3.6, the copy of the special tree $\mathcal{S}$ in $\mathcal{C}$ would imply that $\mathcal{C}$ is special. However, by $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{S}$, the tree $\mathcal{T} \cup S$ cannot be special, and so $\mathcal{C}$ must be nonspecial. $\square$

The proper forcing axiom (PFA) implies that any two Aronszajn trees $\mathcal{T}, \mathcal{T}' \subseteq {}^{<\omega_1}\omega$ are *club-isomorphic*. That is, for some club $E \subseteq \omega_1$,

- $(\{x \in \mathcal{T} \mid \mathrm{dom}(x) \in E\}, \subset)$, and
- $(\{x \in \mathcal{T}' \mid \mathrm{dom}(x) \in E\}, \subset)$

are order-isomorphic.

Problem 3.9. *Does every special Aronszajn tree club-isomorphic to a coherent one?*