Definition 4.1 (Maximal weight)**.** Define $\rho_1 : [\omega_1]^2 \to \omega$ by letting for all $\beta < \delta < \omega_1$:

$$\rho_1(\beta, \delta) := \max\{|C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta| \mid i < \rho_2(\beta, \delta)\}.$$ [1]

For every $\delta < \omega_1$, derive the fiber map $\rho_{1\delta} : \delta \to \omega$ via $\rho_{1\delta}(\beta) := \rho_1(\beta, \delta)$. For simplicity, we do consider $\rho_{1\delta}(\beta)$ for $\beta = \delta$, setting $\rho_{1\delta}(\delta) := 0$. For all $n < \omega$, write

- $D_n(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) = n\}$, and
- $D_{\leq n}(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) \leq n\}$.

Lemma 4.2. *For every $\delta < \omega_1$, $\rho_{1\delta}$ is finite-to-one.*
Equivalently, $D_n(\delta)$ and $D_{\leq n}(\delta)$ are finite for all $\delta < \omega_1$ and $n < \omega$.

Proof. Suppose not, and let $\delta < \omega_1$ be the least counterexample.

Pick $n < \omega$ for which $D_{\leq n}(\delta)$ infinite. As every $\alpha \in D_{\leq n}(\delta)$ satisfies $|C_\delta \cap \alpha| \leq \rho_{1\delta}(\alpha) \leq n$, we may define $f : D_{\leq n}(\delta) \to n+1$ by stipulating

$$f(\alpha) := |C_\delta \cap \alpha|.$$

Since we assume that $D_{\leq n}(\delta)$ is infinite, let us fix a subset $X \subseteq D_{\leq n}(\delta)$ of order-type ω on which f is constant. In particular, $\min(C_\delta \setminus \alpha_1) = \min(C_\delta \setminus \alpha_2)$ for all $\alpha_1, \alpha_2 \in X$, say it is δ'. Then $X \in [\delta']^\omega$, and so by $\delta' < \delta$ and minimality of the latter, we may find some $\alpha' \in X$ such that $\rho_{1\delta'}(\alpha') > n$. That is, $\alpha' \in X \setminus D_{\leq n}(\delta')$. By $\min(C_\delta \setminus \alpha') = \delta'$, we have $\mathrm{tr}(\alpha', \delta) = \langle\delta\rangle^\frown \mathrm{tr}(\alpha', \delta')$, and hence

$$\rho_{1\delta}(\alpha') = \max\{|C_\delta \cap \alpha'|, \rho_{1\delta'}(\alpha')\} > n,$$

contradicting the fact that $\alpha' \in X \subseteq D_{\leq n}(\delta)$. $\square$

Consider the tree $\mathcal{T}(\rho_1) := \{\rho_{1\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$.

Lemma 4.3. *$\mathcal{T}(\rho_1)$ is coherent.*

Proof. Suppose not. Then there exist $\gamma < \delta < \omega_1$ for which $\{\alpha < \gamma \mid \rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)\}$ is infinite. Let $\delta < \omega_1$ be the least for which there exists $\gamma < \delta$ and a subset $X \subseteq \gamma$ of order-type ω with $\rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)$ for all $\alpha \in X$. Put $\beta := \sup(X)$, $\beta^+ := \min(C_\delta \setminus \beta)$ and $\beta^- := \sup(C_\delta \cap \beta)$. By $\beta \leq \gamma < \delta$, $C_\delta \cap \beta$ is finite, and so $\mathrm{cf}(\beta) = \omega$ entails

$$\beta^- < \beta \leq \beta^+ < \delta.$$

Put $n := |C_\delta \cap \beta|$, and then $X' := \{\alpha \in X \setminus (\beta^- + 1) \mid \rho_{1\delta}(\alpha) > n\}$. By Lemma 4.2, we have $\mathrm{otp}(X') = \omega$, so that $\sup(X') = \sup(X) = \beta$. By minimality of δ, we get from $\beta \leq \max\{\gamma, \beta^+\} < \delta$ that there exists $\alpha \in X'$ such that $\rho_{1\gamma}(\alpha) = \rho_{1\beta^+}(\alpha)$.

[1] This is indeed quite similar to the function $\lambda(\beta, \delta) := \max\{\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) \mid i < \rho_2(\beta, \delta)\}$ from Lecture #2.

By $\beta^- < \alpha < \beta \leq \beta^+$, we have:

(1) $\min(C_\delta \setminus \alpha) = \min(C_\delta \setminus \beta) = \beta^+$, so that $\mathrm{tr}(\alpha, \delta) = \langle \delta \rangle ^\frown \mathrm{tr}(\alpha, \beta^+)$;
(2) $|C_\delta \cap \alpha| = |C_\delta \cap \beta| = n < \rho_{1\delta}(\alpha)$, so that $\rho_{1\delta}(\alpha) = \max\{|C_\delta \cap \alpha|, \rho_{1\beta^+}(\alpha)\} = \rho_{1\beta^+}(\alpha)$.

It follows that $\rho_{1\delta}(\alpha) = \rho_{1\beta^+}(\alpha) = \rho_{1\gamma}(\alpha)$, contradicting the fact that $\alpha \in X$. □

Corollary 4.4. *$\mathcal{T}(\rho_1)$ is a coherent Aronszajn tree.*

Proof. By Lemma 4.3, for every $\beta < \omega_1$, any element of the β^{th}-level of $\mathcal{T}(\rho_1)$ is a function from β to ω that differs from $\rho_{1\beta}$ on a finite set, and hence all levels of $\mathcal{T}(\rho_1)$ are countable.

Finally, if $b : \omega_1 \to \omega$ is a cofinal branch through $\mathcal{T}(\rho_1)$, then by the pigeonhole principle, we may fix an uncountable $X \subseteq \omega_1$ on which b is constant, with value, say, k. Pick $\beta < \omega_1$ such that $X \cap \beta$ is infinite. By $b \restriction \beta \in \mathcal{T}(\rho_1)$, let us pick $\delta < \omega_1$ such that $b \restriction \beta = \rho_{1\delta} \restriction \beta$. Then $D_k(\delta)$ covers the infinite set $X \cap \beta$, contradicting Lemma 4.2. □

Definition 4.5. An ω_1-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is said to be *normal* if for all $x \in T$ and $\beta < \omega_1$, there exists some $y \in \mathcal{T}_\beta$ which is compatible with x.

Exercise 4.6. *Prove that for all $\gamma < \delta < \omega_1$, if $\gamma = \min(C_\delta)$, then $\rho_{1\gamma} \subseteq \rho_{1\delta}$. Conclude that for an appropriate choice of a ladder system (in ZFC), $\mathcal{T}(\rho_1)$ is moreover normal.*

Exercise 4.7. *Prove that any normal Aronszajn tree is splitting. That is, for any node x in such a tree, there are nodes y, z extending it such that y and z are incompatible.*

Coming back to Corollary 4.4, it is also possible to prove that $\mathcal{T}(\rho_1)$ does not admit a cofinal branch by showing that ρ_1 enjoys an unboundedness feature similar to that of ρ_2.[2] We use this opportunity to present Lázár's *free set lemma.*

Lemma 4.8 (Lázár, 1936)**.** *For every function $f : \omega_1 \to [\omega_1]^{<\omega}$, there exists some uncountable $A \subseteq \omega_1$ such that $\alpha \notin f(\beta)$ and $\beta \notin f(\alpha)$ for all two distinct α, β from A.*

Proof. Given f as above, derive $g : \omega_1 \to \omega_1$ and $h : \omega_1 \to \omega_1$ by stipulating $g(\alpha) := \sup(f(\alpha) \cap \alpha)$ and $h(\alpha) := \sup(f(\alpha)) + 1$. Consider the club $C := \{\alpha \mid h[\alpha] \subseteq \alpha\}$. By Fodor's lemma, there exists a stationary $S \subseteq \{\alpha \in C \mid \alpha \text{ is a limit nonzero ordinal}\}$, such that $g \restriction S$

[2] Recall Lecture #2.

is constant, with some value, say, ε.[3] We claim that $A := S \setminus (\varepsilon + 1)$ is as sought. To see this, let $\alpha < \beta$ be arbitrary elements of A.

▶ If $\alpha \in f(\beta)$, then $\alpha \in f(\beta) \cap \beta$, so that $\alpha < g(\beta) = \varepsilon$. However $\alpha > \varepsilon$.

▶ If $\beta \in f(\alpha)$, then $\beta \in h(\alpha)$. However $\alpha < \beta$ and $\beta \in C$, so that $h(\alpha) < \beta$. □

Corollary 4.9 (Unboundedness). *Suppose that X is an uncountable family of pairwise disjoint finite subsets of ω_1. For all $k < \omega$, there exists an uncountable $Y \subseteq X$ in which $\min(\rho_1[a \times b]) \geq k$ for all $a < b$ from Y.*

Proof. Since X consists of pairwise disjoint finite sets, we can find some $<$-increasing sequence of elements of X, $\langle x_\alpha \mid \alpha < \omega_1 \rangle$. Fix an arbitrary $k < \omega$. Define $f : \omega_1 \to [\omega_1]^{<\omega}$ by stipulating:

$$f(\beta) := \{\alpha < \omega_1 \mid \exists \delta \in x_\beta [x_\alpha \cap D_{\leq k}(\delta) \neq \emptyset]\}.$$

Since $D_{\leq k}(\delta)$ is finite for all δ in the finite set x_β, and since the elements of $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ are pairwise disjoint, $f(\beta)$ is indeed finite.

Let $A \subseteq \omega_1$ be some uncountable f-free set. Trivially, $Y := \{x_\beta \mid \beta \in A\}$ is uncountable. To see that Y is as sought, let $a < b$ be arbitrary sets from Y, and let $(\gamma, \delta) \in a \times b$ be arbitrary. Pick $\alpha < \beta$ such that $a = x_\alpha$ and $b = x_\beta$. Finally, if $\rho_1(\gamma, \delta) < k$, then $\gamma \in x_\alpha \cap D_{\leq k}(\delta)$, so that $\alpha \in f(\beta)$. But this is not the case. □

Utilizing Lemma 4.2, we can now introduce an injective variation of ρ_1.

Definition 4.10. Define $\bar{\rho}_1 : [\omega_1]^2 \to \omega \times \omega$ by letting for all $\beta < \delta < \omega_1$:

$$\bar{\rho}_1(\beta, \delta) := (\rho_{1\delta}(\beta), |D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta|).$$

That is, $\bar{\rho}_1(\beta, \delta) = (\rho_{1\delta}(\beta), |\{\alpha < \beta \mid \rho_{1\delta}(\alpha) = \rho_{1\delta}(\beta)\}|)$.

Lemma 4.11. *For all $\delta < \omega_1$, the fiber map $\bar{\rho}_{1\delta} : \delta \to \omega \times \omega$ is injective.*

Proof. Let $\alpha < \beta < \delta$ be arbitrary. If $\rho_{1\delta}(\alpha) \neq \rho_{1\delta}(\beta)$, then clearly $\bar{\rho}_{1\delta}(\alpha) \neq \bar{\rho}_{1\delta}(\beta)$. Now, suppose that $\rho_{1\delta}(\alpha) = \rho_{1\delta}(\beta)$, say it is equal to n. Then:

- $(D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta) \cap \alpha = D_n(\delta) \cap \alpha = D_{\rho_{1\delta}(\alpha)}(\delta) \cap \alpha$, and
- $\alpha \in D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta$.

So $D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta$ is a proper end-extension of the finite set $D_{\rho_{1\delta}(\beta)}(\delta) \cap \alpha$, and hence their cardinality is not the same. Consequently, $\bar{\rho}_{1\delta}(\alpha) \neq \bar{\rho}_{1\delta}(\beta)$. □

[3]Note that g is regressive due to the fact that $f(\alpha) \cap \alpha$ is a finite — and hence bounded — subset of α, for all limit nonzero $\alpha < \omega_1$.

We have also maintained coherence:

Lemma 4.12. *For every $\gamma < \delta < \omega_1$, the set $\{\beta < \gamma \mid \bar\rho_{1\gamma}(\beta) \neq \bar\rho_{1\delta}(\beta)\}$ is finite.*

Proof. Let $\gamma < \delta < \omega_1$ be arbitrary. Consider the finite set

$$X := \{\alpha < \gamma \mid \rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)\}.$$

Let $n := \max\{0, \rho_{1\gamma}(\alpha), \rho_{1\delta}(\alpha) \mid \alpha \in X\}$, and then consider the finite set $Y := D_{\leq n}(\gamma) \cup D_{\leq n}(\delta)$. Note that $X \subseteq Y$. We claim that

$$\{\beta < \gamma \mid \bar\rho_{1\gamma}(\beta) \neq \bar\rho_{1\delta}(\beta)\} \subseteq Y.$$

To see this, let $\beta \in \gamma \setminus Y$ be arbitrary.

► By $X \subseteq Y$, we have $\beta \notin X$, and hence $\rho_{1\gamma}(\beta) = \rho_{1\delta}(\beta)$.

► By $\beta \notin Y$, there exists an integer $m > n$ such that $\rho_{1\gamma}(\beta) = \rho_{1\delta}(\beta) = m$, and hence it suffices to show that $D_m(\gamma) \cap \beta = D_m(\delta) \cap \beta$. And indeed, we moreover have $D_m(\gamma) = D_m(\delta) \cap \gamma$, simply because $D_m(\gamma) \cap X = \emptyset = D_m(\delta) \cap X$ as a consequence of $m > n$. □

Corollary 4.13. *$\mathcal{T}(\bar\rho_1) := \{\bar\rho_{1\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$ is a coherent Aronszajn tree consisting of injections!*

Exercise 4.14. *Show that $\mathcal{T}(\rho_1)$ is special iff $\mathcal{T}(\bar\rho_1)$ is special.*

Recall that a *Souslin tree* is an Aronszajn tree that does not admit an uncountable antichain. In a paper from 1984, Shelah proved that adding a Cohen real introduces a Souslin tree. A relatively simple proof of this fact goes as follows.

Exercise 4.15 (Todorcevic, 1987)**.** *If $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is a coherent Aronszajn tree consisting of injections, and $c : \omega \to \omega$ is a Cohen real, then $\{c \circ t \mid t \in \mathcal{T}\}$ is a coherent Souslin tree.*

Exercise 4.16. *Prove that Souslin trees are not $\mathbb{R}$-embeddable.*

Lemma 4.17. *$\mathcal{T}(\rho_1)$ and $\mathcal{T}(\bar\rho_1)$ are $\mathbb{R}$-embeddable.*

Proof. Define $f : \mathcal{T}(\rho_1) \to \mathcal{T}(\bar\rho_1)$, as follows. Given a finite-to-one function $t : \gamma \to \omega$, define $f(t) : \gamma \to \omega \times \omega$ by letting for all $\beta < \gamma$:

$$f(t)(\beta) := (t(\beta), |\{\alpha < \beta \mid t(\alpha) = t(\beta)|).$$

It is clear that f is an order-preserving map from $(\mathcal{T}(\rho_1), \subset)$ to $(\mathcal{T}(\bar\rho_1), \subset)$. Next, fix an injection $\pi : \omega \times \omega \to \omega$,[4] and define $g : \mathcal{T}(\bar\rho_1) \to \mathcal{P}(\omega)$ by stipulating that $g(t) := \pi[\text{range}(t)]$. As elements of $\mathcal{T}(\bar\rho_1)$ are injections, we get that f is order-preserving map from $(\mathcal{T}(\bar\rho_1), \subset)$ to $(\mathcal{P}(\omega), \subset)$.

[4] e.g., $\pi(a, b) := 2^a 3^b$.

Finally, define $h : \mathcal{P}(\omega) \rightarrow [0,1]$ by stipulating that $g(A) := \sum_{n \in A} \frac{1}{2^{n+1}}$. Evidently, for all $A \subset B \subseteq \omega$, we have

$$g(B) = g(A) + g(B \setminus A) > g(A),$$

so that h is an order-preserving map from $(\mathcal{P}(\omega), \subset)$ to the real line $(\mathbb{R}, <)$. It follows that $h \circ g \circ f$ witnesses that $\mathcal{T}(\rho_1)$ is $\mathbb{R}$-embeddable, and $h \circ g$ witnesses that $\mathcal{T}(\bar{\rho}_1)$ is $\mathbb{R}$-embeddable. $\square$

Is $\mathcal{T}(\rho_1)$ is special? This is surely, consistently, the case, as it is consistent with ZFC that *all* Aronszajn trees are special. However, the following is Question 2.2.18 from Stevo's book.

Problem 4.18 (Todorcevic)**.** *What is the condition one needs to put on a given ladder system in order to guarantee that the corresponding tree $\mathcal{T}(\rho_1)$ be special?*

Next week, we shall show that it is also consistent that for some choice of a ladder system, the corresponding tree $\mathcal{T}(\rho_1)$ be nonspecial. This will give a consistent example of an Aronszajn tree which is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.

Problem 4.19. *Must there exist a ladder system for which the corresponding tree $\mathcal{T}(\rho_1)$ be special?*

Problem 4.20. *Suspose that there exists a nonspecial Aronszajn tree. Must there exist a ladder system for which the corresponding tree $\mathcal{T}(\rho_1)$ be nonspecial?*[5]

[5] Recall that by the definition given in Lecture #1, all ladders are required to be of order-type $\leq \omega$.