Exercise 5.1. *Refute: for every $\gamma < \delta < \omega_1$, the set $\{\alpha < \gamma \mid \rho_{1\gamma}(\alpha) = \rho_{1\delta}(\alpha)\}$ is closed.*

Write $A \sqsubseteq B$ iff there exists some β such that $A = B \cap \beta$. Now, consider the following principle.

Definition 5.2 (Hrušák and Martínez-Ranero)**.** $\star_0$ asserts the existence of a ladder system $\langle C_\delta \mid \delta < \omega_1 \rangle$ such that:

- for every $\delta < \omega_1$, $C_{\delta+1} = \{\delta\}$;
- for every limit $\delta < \omega_1$, C_δ is some cofinal subset of δ of order-type ω;
- for every function $f : \omega_1 \to \omega$, there exist limit ordinals $\gamma < \delta < \omega_1$ with $f(\gamma) = f(\delta)$ such that $\gamma \in C_\delta$ and $C_\delta \cap \gamma \sqsubseteq C_\gamma$.

Exercise 5.3 (Hrušák and Martínez-Ranero)**.** *Show that $\star_0$ holds after forcing to add a Cohen real.*

Conclude that $\star_0$ is consistent with Martin's Axiom for σ-centered posets.

The next theorem implies, in particular, that $\star_0$ is inconsistent with Martin's Axiom.

Theorem 5.4 (Hrušák and Martínez-Ranero, 2005)**.** *Suppose that ρ_1 is derived from a walk along a $\star_0$-sequence. Then $\mathcal{T}(\rho_1)$ is nonspecial.*

Proof. Suppose that $\{A_n \mid n < \omega\}$ is a cover of $\mathcal{T}(\rho_1)$ by antichains. Define $f : \omega_1 \to \omega$ by stipulating $f(\delta) := \min\{n < \omega \mid \rho_{1\delta} \in A_n\}$. Now, pick limit ordinals $\gamma < \delta < \omega_1$ with $f(\gamma) = f(\delta)$ such that $\gamma \in C_\delta$ and $C_\delta \cap \gamma \sqsubseteq C_\gamma$. We shall obtain a contradiction by showing that $\rho_{1\gamma}, \rho_{1\delta}$ (that belong to the same antichain) are actually compatible.

Let $\alpha < \gamma$ be arbitrary. We shall show that $\rho_{1\gamma}(\alpha) = \rho_{1\delta}(\alpha)$.

Let $\{\delta_i \mid i < \omega\}$ denote the increasing enumeration of C_δ, and $\{\gamma_i \mid i < \omega\}$ denote the increasing enumeration of C_γ. Let $n = |C_\delta \cap \gamma|$. By $C_\delta \cap \gamma \sqsubseteq C_\gamma$, we have $\delta_i = \gamma_i$ for all $i < n$, and $\delta_n = \gamma$.

► If there exists some $i < n$ such that $\alpha \leq \delta_i$, then for the least such i, we have

- $\mathrm{tr}(\alpha, \delta) = \langle \delta \rangle^\frown \mathrm{tr}(\alpha, \delta_i)$,
- $\mathrm{tr}(\alpha, \gamma) = \langle \gamma \rangle^\frown \mathrm{tr}(\alpha, \delta_i)$,

and

$$\rho_{1\gamma}(\alpha) = \max\{|C_\gamma \cap \alpha|, \rho_{1\delta_i}(\alpha)\} = \max\{|C_\delta \cap \gamma \cap \alpha|, \rho_{1\delta_i}(\alpha)\} = \max\{|C_\delta \cap \alpha|, \rho_{1\delta_i}(\alpha)\} = \rho_{1\delta}(\alpha).$$

► If there exists no $i < n$ such that $\alpha < \delta_i$, then by $n = |C_\delta \cap \gamma|$, this means that $\min(C_\delta \setminus \alpha) = \delta_n = \gamma$. Consequently, $\mathrm{tr}(\alpha, \delta) =$

$\langle\delta\rangle^\frown \mathrm{tr}(\alpha,\gamma)$, so that $\rho_{1\delta}(\alpha) = \max\{|C_\delta \cap \alpha|, \rho_{1\gamma}(\alpha)\}$. By definition, we have $\rho_{1\gamma}(\alpha) \geq |C_\gamma \cap \alpha|$. As $C_\gamma \supseteq C_\delta \cap \gamma$, we have $|C_\gamma \cap \alpha| \geq |C_\delta \cap \gamma \cap \alpha| = |C_\delta \cap \alpha|$, and then $\rho_{1\delta}(\alpha) = \rho_{1\gamma}(\alpha)$. □

Definition 5.5. The principle $\diamondsuit$ asserts the existence of a sequence $\langle X_\alpha \mid \alpha < \omega_1\rangle$ such that for every $X \subseteq \omega_1$, there exists a nonzero limit ordinal such that $X_\alpha = X \cap \alpha$.

Exercise 5.6. *Show that $\diamondsuit$ implies that $2^{\aleph_0} = \aleph_1$.*

Exercise 5.7 (P. Larson)**.** *Show that $\diamondsuit$ entails $\star_0$.*

Corollary 5.8. *$\diamondsuit$ entails the existence of an ω_1-tree which is $\mathbb{R}$-embeddable, but not $\mathbb{Q}$-embeddable.*

Definition 5.9. The principle $\clubsuit$ asserts the existence of a ladder system $\langle C_\alpha \mid \alpha < \omega_1\rangle$ such that for every uncountable $X \subseteq \omega_1$, there exists a nonzero limit ordinal $\alpha < \omega_1$ such that $C_\alpha \subseteq X$.

Exercise 5.10 (Devlin)**.** *Show that $\diamondsuit$ holds iff ($2^{\aleph_0} = \aleph_1$ and $\clubsuit$ holds).*

Problem 5.11. *Does $\clubsuit$ entail $\star_0$?*

In today's lecture, we shall consider abstract forms of ϱ functions.

Definition 5.12. For infinite cardinals λ, χ, a function $\varrho : [\lambda^+]^2 \to \lambda$ is said to be:[1]

- *locally small*, if $|\{\alpha < \beta \mid \varrho_\beta(\alpha) \leq \nu\}| < \lambda$ for all $\nu < \lambda$ and $\beta < \lambda^+$;
- *χ-coherent*, if $|\{\alpha < \beta \mid \varrho_\beta(\alpha) \neq \varrho_\gamma(\alpha)\}| < \chi$ for all $\beta < \gamma < \lambda^+$.

In the previous lecture, we saw that $\rho_1 : [\omega_1]^2 \to \omega$ is locally small and ω-coherent.

Theorem 5.13 (B. König, 2003)**.** *There exists no $\varrho : [\omega_2]^2 \to \omega_1$ which is locally small and ω-coherent.*

Proof. Towards a contradiction, suppose that $\varrho : [\omega_2]^2 \to \omega_1$ is a counterexample. Since ϱ is locally small, we get that for every $\nu < \omega_1$, $D_{\leq\nu}(\delta) := \{\alpha < \delta \mid \varrho_\delta(\alpha) \leq \nu\}$ is countable.

For every ordinal $\delta < \omega_2$, let us denote $\delta^* := \delta + \omega_1$.

Claim 5.13.1. *For every $\delta < \omega_2$, there exists some $\nu < \omega_1$ such that:*

$$\mathrm{otp}(D_{\leq\nu}(\delta)) + \omega < \mathrm{otp}(D_{\leq\nu}(\delta^*)).$$

[1] As usual, $\varrho_\delta : \delta \to \lambda$ denotes the fiber map satisfying $\varrho_\delta(\alpha) = \varrho(\alpha,\delta)$ for all $\alpha < \delta$.

Proof. Since $\mathsf{d}(\delta,\delta^*) := \{\alpha < \delta \mid \varrho_\delta(\alpha) \neq \varrho_{\delta^*}(\alpha)\}$ is finite, $\nu_0 := \sup\{\varrho_\delta(\alpha), \varrho_{\delta^*}(\alpha) \mid \alpha \in \mathsf{d}(\delta,\delta^*)\}$ is a countable ordinal.

Since the ordinal-interval $[\delta, \delta+\omega+1]$ is countable, we have that $\nu_1 := \sup(\varrho_{\delta^*}``[\delta,\delta+\omega+1])$ is yet another countable ordinal.

Put $\nu := \max\{\nu_0,\nu_1\}+1$. Then:

▶ By $\nu > \nu_0$, we have $D_{\le\nu}(\delta) = D_{\le\nu}(\delta^*) \cap \delta$.

▶ By $\nu > \nu_1$, we have $[\delta,\delta+\omega+1] \subseteq D_{\le\nu}(\delta^*) \cap [\delta,\delta^*)$, so that $\omega < \operatorname{otp}(D_{\le\nu}(\delta^*) \cap [\delta,\delta^*))$.

Altogether, $\operatorname{otp}(D_{\le\nu}(\delta)) + \omega < \operatorname{otp}(D_{\le\nu}(\delta^*))$. □

Consider the set $E^{\omega_2}_{\omega_1} := \{\delta < \omega_2 \mid \operatorname{cf}(\delta) = \omega_1\}$. For each $\delta \in E^{\omega_1}_{\omega_1}$, let $\nu(\delta)$ be given by the preceding claim. By the pigeonhole principle, let us fix some $\nu < \omega_1$, and $A \subseteq E^{\omega_2}_{\omega_1}$ of size $\aleph_2$ for which $\nu(\delta) = \nu$ for all $\delta \in A$.

The next claim is really where we use the fact that ϱ is ω-coherent.

Claim 5.13.2. *For every $\gamma \le \delta < \omega_2$:*

$$\operatorname{otp}(D_{\le\nu}(\gamma)) < \operatorname{otp}(D_{\le\nu}(\delta)) + \omega.$$

Proof. Since $\mathsf{d}(\gamma,\delta) := \{\alpha < \gamma \mid \varrho_\gamma(\alpha) \neq \varrho_\delta(\alpha)\}$ is a finite set, there exists some integer $z \in \mathbb{Z}$ such that

$$\operatorname{otp}(D_{\le\nu}(\gamma)) = \operatorname{otp}(D_{\le\nu}(\delta)\cap\gamma)+z \le \operatorname{otp}(D_{\le\nu}(\delta))+z < \operatorname{otp}(D_{\le\nu}(\delta))+\omega.$$

□

Define $f : A \to \omega_1$ by stipulating:

$$f(\delta) := \operatorname{otp}(D_{\le\nu}(\delta^*)).$$

Claim 5.13.3. *$f(\beta) < f(\delta)$ for all $\beta < \delta$ both from A.*

Proof. Let $\beta < \delta$ be arbitrary elements of A. Denote $\gamma := \beta^*$. Since $\delta \in E^{\omega_2}_{\omega_1}$ and since γ is the first element of $E^{\omega_2}_{\omega_1}$ above β, we have $\beta < \gamma \le \delta$, so that

$$f(\beta) = \operatorname{otp}(D_{\le\nu}(\gamma)) < \operatorname{otp}(D_{\le\nu}(\delta)) + \omega < \operatorname{otp}(D_{\le\nu}(\delta^*)) = f(\delta).$$

□

In particular, f forms an injection from A to ω_1, contradicting the fact that $|A| = \aleph_2$. □

The preceding theorem raises the following question. Does there exist a function $\varrho : [\omega_2]^2 \to \omega_1$ which is locally small and ω_1-coherent?

It turns out that the answer is affirmative, and moreover, for every infinite regular cardinal λ, for a suitable choice of a ladder system over λ^+, the corresponding function $\rho_1 : [\lambda^+]^2 \to \lambda$ is indeed locally small and λ-coherent. Out next goal is to show that, in general, it is hopeless to expect that this will generalize to λ singular. To prove this, let us first recall some basic stuff around ultrafilters and strongly compact cardinals.

Definition 5.14. A filter $\mathcal{F}$ over a set X is a collection satisfying the following:

- $\mathcal{F} \subseteq \mathcal{P}(X)$;
- $X \in \mathcal{F}$, while $\emptyset \notin \mathcal{F}$;
- if $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An illuminating example of a filter is the following

$$\mathcal{F}_1 := \{A \subseteq \mathbb{R} \mid \mathbb{R} \setminus A \text{ is a null set}\}.$$[2]

Definition 5.15. A filter $\mathcal{F}$ is said to be *κ-complete* if for any $\{A_i \mid i < \sigma\} \subseteq \mathcal{F}$ with $\sigma < \kappa$, we have $\bigcap_{i<\sigma} A_i \in \mathcal{F}$.

For instance, $\mathcal{F}_1$ is $\aleph_1$-complete, since the countable union of null sets is null.

Exercise 5.16. *Prove that any filter is $\aleph_0$-complete.*

Definition 5.17. A filter $\mathcal{F}$ is said to be uniform if $|A| = |B|$ for all $A, B \in \mathcal{F}$.

For instance, for any infinite cardinal θ, $\{X \subseteq \theta \mid |\theta \setminus X| < \theta\}$ is uniform and $\mathrm{cf}(\theta)$-complete.

Definition 5.18. A filter $\mathcal{F}$ over some set X is said to be an *ultrafilter*, if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $(X \setminus A) \in \mathcal{F}$.

Lemma 5.19 (Pigeonhole principle for ultrafilters)**.** *Suppose that $\mathcal{U}$ is a κ-complete ultrafilter.*

For any $A \in \mathcal{U}$, and any function $f : A \to \sigma$ with $\sigma < \kappa$, there exists some $B \subseteq A$ with $B \in \mathcal{U}$ on which f is constant.

Proof. For each $i < \sigma$, let $A_i := \{a \in A \mid f(a) = i\}$.

If there exists some $i < \sigma$ such that $A_i \in \mathcal{U}$, then we are done. Otherwise, since $\mathcal{U}$ is an ultrafilter, we infer that $A \setminus A_i$ is in $\mathcal{U}$ for all $i < \sigma$. Since $\mathcal{U}$ is κ-complete, then, $\emptyset = A \setminus \bigcup_{i<\sigma} A_i = \bigcap_{i<\sigma}(A \setminus A_i)$ is in $\mathcal{U}$. This is a contradiction. □

By Zorn's lemma, any filter $\mathcal{F}$ over a set X may be extended to an ultrafilter $\mathcal{U}$ over X, so that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{P}(X)$. However, in general, the ultrafilter $\mathcal{U}$ may have smaller completeness degree than that of $\mathcal{F}$.

[2]Recall that $A \subseteq \mathbb{R}$ is said to be a *null set* if for every $\epsilon > 0$, there exists a countable sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $A \subseteq \bigcup_{n<\omega} I_n$, and the sum $\sum_{n<\omega} \mathrm{Diam}(I_n)$ of the diameters of the I_n's is $< \epsilon$.

Definition 5.20. An uncountable cardinal κ is said to be *strongly compact* if for every set X, any κ-complete filter $\mathcal{F}$ over X may be extended to a κ-complete ultrafilter $\mathcal{U}$ over X.

Exercise 5.21. *If κ is strongly compact, then κ is a strong inaccessible.*

Exercise 5.22. *If κ is strongly compact, then for every set X with $\mathrm{cf}(|X|) \geq \kappa$, there exists a uniform κ-complete ultrafilter over X.*

Definition 5.23. A coloring $c : [\theta]^2 \to \theta$ is said to *subadditive* if the below two hold:

- *subadditivity of the first kind.* That is, for all $\alpha \leq \beta \leq \gamma < \theta$:
$$c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\};$$
- *subadditivity of the second kind.* That is, for all $\alpha \leq \beta \leq \gamma < \theta$:
$$c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}.$$

Note that subadditivity is equivalent to a coherence property of the derived filtration:

Exercise 5.24. *Given $c : [\theta]^2 \to \theta$, let us write $D_{\leq\nu}(\gamma) := \{\alpha < \gamma \mid c(\alpha, \gamma) \leq \nu\}$ for all $\nu, \gamma < \theta$. Then the following are equivalent:*

(1) *c is subadditive;*
(2) *for all $\beta < \gamma < \theta$ and $\nu < \theta$, if $\beta \in D_{\leq\nu}(\gamma)$, then*
$$D_{\leq\nu}(\beta) = D_{\leq\nu}(\gamma) \cap \beta.$$

Exercise 5.25. *Suppose that $c : [\theta]^2 \to \theta$ is subadditive.*

For all $\alpha < \beta < \gamma < \theta$, if $c(\alpha, \beta) > c(\beta, \gamma)$, then $c(\alpha, \gamma) = c(\alpha, \beta)$.

Lemma 5.26. *Suppose that κ is strongly compact, σ is some limit ordinal $< \kappa$, and θ is some regular cardinal $\geq \kappa$.*

For every coloring $c : [\theta]^2 \to \sigma$ which is subadditive of the second kind, there exists $H \in [\theta]^\theta$ such that $\sup(c``[H]^2) < \sigma$.

Proof. Fix a κ-complete uniform ultrafilter $\mathcal{U}$ over θ. Given a coloring c as above, for all $\alpha < \kappa$ and $i < \sigma$, write
$$A^i_\alpha := \{\beta \mid \alpha < \beta < \theta, c(\alpha, \beta) = i\}.$$

Since $\mathcal{U}$ is uniform, $\biguplus_{i<\sigma} A^i_\alpha = \theta \setminus (\alpha + 1)$ is in $\mathcal{U}$. Since $\mathcal{U}$ is a κ-complete ultrafilter, we infer from Lemma 5.19 that there exists some $i_\alpha < \sigma$ such that $A^{i_\alpha}_\alpha \in \mathcal{U}$.

Next, by the (usual) pigeonhole principle, let us fix some $H \in [\theta]^\theta$ and $i < \sigma$ such that $i_\alpha = i$ for all $\alpha \in H$.

We claim that $\sup(c``[H]^2) \leq i$. To see this, let $\alpha < \beta$ be arbitrary elements of H. By $A^{i_\alpha}_\alpha, A^{i_\beta}_\beta \in \mathcal{U}$ and $i_\alpha = i = i_\beta$, we may pick some

$\gamma \in A^i_\alpha \cap A^i_\beta$. Then $c(\alpha,\gamma) = i = c(\beta,\gamma)$, so that subadditivity entails that $c(\alpha,\beta) \leq i$. □

A recursive application of the preceding entails the following.

Exercise 5.27. *Suppose that κ is strongly compact, $\sigma < \kappa \leq \theta = \mathrm{cf}(\theta)$.*

Prove that for every coloring $c : [\theta]^2 \to \sigma$ which is subadditive of the second kind, there exists $H \in [\theta]^\theta$ such that $c``[H]^2$ is finite.

Theorem 5.28. *Suppose that λ is the singular limit of strongly compact cardinals. Then there exists no function $\varrho : [\lambda^+]^2 \to \lambda$ which is simultaneously locally small and λ-coherent.*

Proof. Let $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ be some strictly increasing sequence of strongly-compact cardinals, that converges to λ, and such that $\lambda_0 > \mathrm{cf}(\lambda)$. Utilizing the fact that ϱ is locally small, define $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ by letting for all $\gamma \leq \delta < \lambda^+$:

- $c(\gamma,\delta) := \min\{i < \mathrm{cf}(\lambda) \mid |\mathsf{d}(\gamma,\delta)| \leq \lambda_i\}$, where:
- $\mathsf{d}(\gamma,\delta) := \{\alpha < \gamma \mid \varrho_\gamma(\alpha) \neq \varrho_\delta(\alpha)\}$.

Claim 5.28.1. *c is subadditive.*

Proof. Suppose that $\beta \leq \gamma \leq \delta$.

► To show additivity of the second kind, it suffices to prove that $\mathsf{d}(\beta,\delta) \subseteq \mathsf{d}(\beta,\gamma) \cup \mathsf{d}(\gamma,\delta)$.

Let $\alpha \in \mathsf{d}(\beta,\delta)$ be arbitrary. If $\alpha \notin \mathsf{d}(\beta,\gamma)$, then $\varrho_\beta(\alpha) = \varrho_\gamma(\alpha)$. As $\varrho_\beta(\alpha) \neq \varrho_\delta(\alpha)$, we have $\varrho_\gamma(\alpha) \neq \varrho_\delta(\alpha)$. That is, $\alpha \in \mathsf{d}(\gamma,\delta)$.

► To show additivity of the second kind, it suffices to prove that $\mathsf{d}(\beta,\gamma) \subseteq \mathsf{d}(\beta,\delta) \cup \mathsf{d}(\gamma,\delta)$.

Let $\alpha \in \mathsf{d}(\beta,\gamma)$ be arbitrary. If $\alpha \notin \mathsf{d}(\beta,\delta)$, then $\varrho_\beta(\alpha) = \varrho_\delta(\alpha)$. As $\varrho_\beta(\alpha) \neq \varrho_\gamma(\alpha)$, we have $\varrho_\delta(\alpha) \neq \varrho_\gamma(\alpha)$. That is, $\alpha \in \mathsf{d}(\gamma,\delta)$. □

By Lemma 5.26, let $B \in [\lambda^+]^{\lambda^+}$ be such that $i := \sup(c``[B]^2) + 1$ is $< \mathrm{cf}(\lambda)$.[3] For all $\alpha < \lambda^+$ and $j < \mathrm{cf}(\lambda)$, write

$$B^j_\alpha := \{\beta \in B \setminus (\alpha+1) \mid \varrho_\beta(\alpha) \leq \lambda_j\}.$$

Fix a λ_{i+1}-complete uniform ultrafilter $\mathcal{U}$ over B. By uniformity and $(\mathrm{cf}(\lambda))^+$-completeness of $\mathcal{U}$, for each $\alpha < \lambda^+$, we may pick $j_\alpha < \mathrm{cf}(\lambda)$ such that $B^{j_\alpha}_\alpha \in \mathcal{U}$. By the pigeonhole principle, let us pick $A \in [\lambda^+]^{\lambda^+}$ on which the map $\alpha \mapsto j_\alpha$ is constant, with value, say, j.

By $|A| = |B| = \lambda^+$, let us pick a large enough $\beta \in B$ such that $|A \cap \beta| = \lambda$. Since ϱ is locally small, let us fix $\bar{A} \subseteq A \cap \beta$ of size λ_i such that $\varrho_\beta(\alpha) > \lambda_j$ for all $\alpha \in \bar{A}$.[4]

[3] Notice that this means that $\{\varrho_\beta \mid \beta \in B\}$ is λ_i-coherent.

[4] We can definitely find such an $\bar{A} \subseteq A \cap \beta$ of size λ, but size λ_i is all we need.

Using $(\lambda_i)^+$-completeness and uniformity of $\mathcal{U}$, pick $\beta^* \in \bigcap_{\alpha\in\bar{A}} B^j_\alpha$ above β. Now, for all $\alpha \in \bar{A}$, by $\beta^* \in B^j_\alpha$, we have

$$\varrho_{\beta^*}(\alpha) \leq \lambda_j < \varrho_\beta(\alpha).$$

So, $\bar{A} \subseteq \mathsf{d}(\beta, \beta^*)$ and hence $|\mathsf{d}(\beta, \beta^*)| \geq \lambda_i$, contradicting the fact that $\beta, \beta^* \in B$ and $c(\beta, \beta^*) < i$. □

Exercise 5.29. *Show that for every infinite cardinal λ, there exists a function $\varrho : [\lambda^+]^2 \to \lambda$ which is locally small.*