Given a set I, we let $\mathcal{LO}_I$ denote the set of all linear orderings of I:

$$\mathcal{LO}_I := \{R \in \mathcal{P}(I \times I) \mid R \text{ is a total ordering of } I\}.$$

Note that any linear order $(L, \leq)$ of size κ admits a *representative* in $\mathcal{LO}_\kappa$, by simply taking a bijection $f : L \leftrightarrow \kappa$, and setting $R := \{(f(x), f(y)) \mid x \leq y\}$. So, in some sense, to understand all linear orderings, it suffices to study $\mathcal{LO}_\kappa$ for all cardinals κ.

For $R, S \in \mathcal{LO}_I$, write $R \trianglelefteq S$ if the ordered set (I, R) is embedded in (I, S), that is, if there exists an injection $f : I \to I$ such that $\{(f(x), f(y)) \mid (x, y) \in R\} \subseteq S$. Clearly, $\trianglelefteq$ is pre-order, that is, it is reflexive and transitive. A standard way to turn a pre-order into an order is to define an equivalence relation $\sim$ by letting $R \sim S$ iff $R \trianglelefteq S$ and $S \trianglelefteq R$, and then inherit an ordering on the quotient. However, the above equivalence relation does not express the concept that we are interested in. For instance, the unit interval $[0, 1]$ embeds into the half-open interval $(0, 1]$ via the map $x \mapsto \frac{x+1}{2}$, and $(0, 1]$ embeds into $[0, 1]$ via the identity map, while the representatives of the orderings of $[0, 1]$ and $(0, 1]$ in $\mathcal{LO}_\mathfrak{c}$ cannot be considered equivalent, since one has a minimal element and the other does not. Thus, instead, we shall define the equivalence relation $\sim$ by letting $R \sim S$ iff there exists a bijection $f : I \leftrightarrow I$ such that $S = \{(f(x), f(y)) \mid (x, y) \in R\}$. That is: $f(x)Sf(y)$ iff xRy.

Recall that the *cofinality* of a pre-order $(P, \leq)$ is the least size of a subset $A \subseteq P$ with the property that for all $x \in P$, there exists $y \in A$ with $x \leq y$. The co-initiality of $(P, \leq)$ is defined as the cofinality of $(P, \geq)$, where $x \geq y$ iff $y \leq x$.

Some natural questions that immediately come to mind concerning the structure $(\mathcal{LO}_\kappa, \trianglelefteq)$:

- What is the cofinality of $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What is the co-initiality of $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What are the sizes of maximal chains in $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What are the sizes of maximal antichains in $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- How many equivalence classes are in $(\mathcal{LO}_\kappa, \sim)$?

Exercise 6.1 (Cantor)**.** *Prove that any countable linear order embeds into the rationals* $(\mathbb{Q}, \leq)$.

Corollary 6.2. *The cofinality of* $(\mathcal{LO}_\omega, \trianglelefteq)$ *is* 1.

Definition 6.3. A linear order $(L, \leq)$ is said to be *separable*, if there exists a countable subset $D \subseteq L$, such that for all $a < b$ in L, there exists $d \in D$ with $a \leq d \leq b$.

Cantor's theorem implies:

Exercise 6.4. *A linear order $(L, \preceq)$ is separable iff the real line $(\mathbb{R}, \leq)$ contains a copy of $(L, \preceq)$.*

Exercise 6.5. *$(\omega_1, \in)$ and $(\omega_1, \ni)$ do not contain uncountable separable suborders.*

Exercise 6.6 (Sierpiński, 1933)**.** *A separable linear order does not contain a copy of $(\omega_1, \in)$ or of $(\omega_1, \ni)$.*

Definition 6.7. For a linearly ordered set $(L, \leq)$, write:

- $<:= \{(a,b) \in \leq \mid a \neq b\}$;
- $\leq^* := \{(b,a) \mid (a,b) \in \leq\}$;
- $L^2 := \{(a,b) \mid a \in L, b \in L\}$;
- $[L]^2 := \{(a,b) \in L^2 \mid a < b\}$;[1]
- $\leq^2$ is the relation over L^2 such that $(a,b) \leq^2 (c,d)$ iff $(a \leq c)$ and $(b \leq d)$.

Lemma 6.8. *The co-initiality of $(\mathcal{LO}_\omega, \trianglelefteq)$ is 2.*

Proof. Let $R := \in \restriction \omega^2$ and $S := R^*$, that is, $S = \ni \restriction \omega^2$. Give a $T \in \mathcal{LO}_\omega$, define $c : [\omega]^2 \to 2$ by letting for all $n < m < \omega$: $c(n,m) = 1$ iff $(n,m) \in T$.

By Ramsey's theorem $\omega \to (\omega)^2_2$, we may find some an infinite $H \subseteq \omega$ which is c-homogeneous. Let $f : \omega \leftrightarrow H$ be the order-preserving bijection.

▶ If $c``[H]^2 = \{1\}$, then for all $n < m$, by $f(n) < f(m)$ in H, we have $c(f(n), f(m)) = 1$ and hence $(f(n), f(m)) \in T$. That is, $\{(f(n), f(m)) \mid (n,m) \in R\} \subseteq T$.

▶ If $c``[H]^2 = \{0\}$, then for all $n < m$, by $f(n) < f(m)$ in H, we have $c(f(n), f(m)) = 0$ and hence $(f(n), f(m)) \notin T$, and hence $(f(m), f(n)) \in T$. That is, $\{(f(n), f(m)) \mid (n,m) \in S\} \subseteq T$.

So $\{R, S\}$ is co-initial in $\mathcal{LO}_\omega$. In addition, (ω, S) does not embed in (ω, R) since the latter is well-ordered. Similarly, (ω, R) does not embed in (ω, S). So, the co-initiality of $(\mathcal{LO}_\omega, \trianglelefteq)$ is exactly 2. □

The reader may protest against the involvement of Ramsey's theorem in the preceding proof. However, we will see later that this is no incident, and there is a bilateral relation between Ramsey-type theorems for cardinals κ, and the co-initiality of $(\mathcal{LO}_\kappa, \trianglelefteq)$.

Exercise 6.9. *Exhibit a chain of order-type ω_1 in $(\mathcal{LO}_\omega, \trianglelefteq)$.*

Exercise 6.10. *Prove that $(\mathcal{LO}_\omega, \trianglelefteq)$ has an antichain of cardinality $2^{\aleph_0}$.*

[1] This happens to be equivalent to the set $<$.

Any uncountable subset X of ω_1 satisfies $(X, \in) \cong (\omega_1, \in)$ and $(X, \ni) \cong (\omega_1, \ni)$. In addition, by Exercise 6.4, any uncountable suborder of a separable linear order is a separable linear order. Is there another class of uncountable linear orders such that any uncountable suborder remains inside the class?

Definition 6.11. An uncountable linear order $(L, \leq)$ is said to be a *Countryman line* iff $(L^2, \leq^2)$ may be covered by countably many chains.

Exercise 6.12. *If $(C, \leq)$ is Countryman, then for every uncountable subset X of C, $(X, \leq)$ is Countryman.*

Exercise 6.13. *If $(C, \leq)$ is Countryman, then $(C, \leq^*)$ is Countryman.*

Observation 6.14. *If $(C, \leq)$ is Countryman, then $(C^2, \leq^2)$ is* ccc. *That is, it does not have uncountable antichains.*

Proof. Suppose that $C^2 \subseteq \bigcup_{n<\omega} A_n$, where $(A_n, \leq^2)$ is a chain for all $n < \omega$. If $Y \subseteq C^2$ is an antichain, then $|Y \cap A_n| \leq 1$ for all $n < \omega$, and hence Y is countable. □

Corollary 6.15. *If $(C, \leq)$ is Countryman, then any linear-order that embeds to $(C, \leq)$ and to $(C, \leq^*)$ is countable.*

In particular, if $(\omega_1, \preceq)$ is Countryman, then $\{\preceq, \preceq^*\}$ is an antichain in $\mathcal{LO}_{\omega_1}$.

Proof. Suppose that $(L, \preceq)$ is a linear order, and f is an embedding of $(L, \preceq)$ in $(C, \leq)$, and g is an embedding of $(L, \preceq)$ in $(C, \leq^*)$. Define $h : L \to C^2$ by letting $h(x) := (f(x), g(x))$. Let $x \neq y$ in L be arbitrary. Wlog, $x \prec y$. Then $f(x) < f(y)$ and $g(x) > g(y)$. It follows that range(h) is an antichain in $(C^2, \leq^2)$, and hence countable. As f is injective, so is h, and hence $|L| = |\operatorname{range}(h)| \leq \aleph_0$. □

Lemma 6.16 (Countryman, 1971)**.** *A Countryman line does not contain a copy of uncountable separable linear orders.*

Proof. Suppose that $(C, \leq)$ is a Countryman line, as witnessed by a partition $\{A_n \mid n < \omega\}$ of C^2 into $\leq^2$-chains. Suppose that $(L, \preceq)$ is an uncountable separable linear order that embeds to $(C, \leq)$ via a map $f : L \to C$. Let $D \subseteq L$ be a countable set witnessing that $(L, \preceq)$ is separable.

Claim 6.16.1. *For all $x \in L$, there exist $n < \omega$ and $d \in D$ for which the following holds. There exist $y, z \in L$ such that $y \prec d \prec z$ and $\{(f(x), f(y)), (f(x), f(z))\} \subseteq A_n$.*

Proof. Let $x \in L$ be arbitrary. Since $\{(f(x), f(y)) \mid y \in L\}$ is uncountable, let us pick some $n < \omega$ such that $\{(f(x), f(y)) \mid y \in L\} \cap A_n$ is uncountable. Pick $y_0 \prec y_1 \prec y_2 \prec y_3$ in L such that $(f(x), f(y_i)) \in A_n$ for all $i < 4$. Now, pick $d \in D$ such that $y_1 \preceq d \preceq y_2$. Then n and d works, as witnessed by $y := y_0$ and $z := y_3$. □

For all $x \in L$, pick $(n_x, d_x) \in \omega \times D$ as in the preceding. Since $\omega \times D$ is countable, let us pick $x \prec x'$ in L for which $n_x = n_{x'}$, say n, and $d_x = d_{x'}$, say d. Pick $y \prec d \prec z$ and $y' \prec d \prec z'$ such that

$$\{(f(x), f(y)), (f(x), f(z)), (f(x'), f(y')), (f(x'), f(z'))\} \subseteq A_n.$$

In particular, the set on the left hand side of the above equation is linearly ordered by $\leq^2$. Since $x \prec x'$ and f is order-preserving, we have $f(x) < f(x')$ and hence $f(z) \leq f(y')$. Since f is order-preserving, then $z \preceq y'$, contradicting the fact that $y' \prec d \prec z$. □

Corollary 6.17. *An uncountable separable linear order does not contain a copy of a Countryman line.*

Proof. Suppose not. By Exercise 6.4, this means that the real line $(\mathbb{R}, \leq)$ contains a copy of a Countryman line $C \subseteq \mathbb{R}$. Then, by Exercise 6.4 again, this means that $(C, \leq)$ is a separable Countryman line, contradicting Lemma 6.16. □

In his 1971 paper, Roger Countryman conjectured that the separability hypothesis in Lemma 6.16 is surplus, i.e., that the now-called *Countryman lines* simply do not exist. However, in a paper from 1976, Shelah introduced a construction of such a linear order. We shall soon present a construction that uses walks on ordinals.

Lemma 6.18. *If $(C, \leq)$ is Countryman, then it does not contain a copy of $(\omega_1, \in)$.*

Proof. Let $\{A_n \mid n < \omega\}$ witness that $(C, \leq)$ is Countryman. Suppose that $f : \omega_1 \to C$ is an order-preserving function from $(\omega_1, \in)$ to $(C, <)$. For all $\alpha < \omega_1$, fix some $n_\alpha < \omega$ such that the set $\{(f(\alpha), f(\beta)) \mid \alpha < \beta < \omega_1\} \cap A_n$ is uncountable. By the pigeonhole principle, let us fix two ordinals $\alpha < \alpha' < \omega_1$ such that $n_\alpha = n_{\alpha'}$, say, it is n. Let:

- $B = \{\beta < \omega_1 \mid (f(\alpha), f(\beta)) \in A_n\}$;
- $B' = \{\beta' < \omega_1 \mid (f(\alpha'), f(\beta')) \in A_n\}$.

Then B and B' are uncountable, and we may find uncountable subsets $\bar{B} \subseteq B$ and $\bar{B}' \subseteq B'$ such that $\bar{B} \cap \bar{B}' = \emptyset$.

For all $\beta \in \bar{B}$ and $\beta' \in \bar{B}'$, we have $\{(f(\alpha), f(\beta)), (f(\alpha'), f(\beta'))\} \subseteq A_n$, and so by $\alpha \in \alpha'$ we have $f(\alpha) < f(\alpha')$ and hence $f(\beta) \leq f(\beta')$ and hence $\beta \in \beta'$. So $\mathrm{otp}(\bar{B} \cup \bar{B}', \in) \geq \omega_1 + \omega_1$, contradicting the fact that $\bar{B} \cup \bar{B}' \subseteq \omega_1$. □

Corollary 6.19. $(\omega_1, \in)$ *does not contain a copy of a Countryman line.*

Proof. If L is an uncountable subset of ω_1 such that $(L, \in)$ is order-isomorphic to a Countryman line, then by $\mathrm{otp}(L, \in) = \omega_1$, this would mean that $(\omega_1, \in)$ itself is a Countryman line. However, the previous Lemma tells us that a Countryman line cannot contain a copy of $(\omega_1, \in)$. □

Exercise 6.20. (1) *If* $(C, \leq)$ *is Countryman, then it does not contain a copy of* $(\omega_1, \ni)$.
(2) $(\omega_1, \ni)$ *does not contain a copy of a Countryman line.*

Definition 6.21. Fix an arbitrary injection $f : \omega_1 \to \mathbb{R}$, and let $\leq_{\mathbb{R}} := \{(\alpha, \beta) \in (\omega_1)^2 \mid f(\alpha) \leq f(\beta)\}$. Then $\leq_{\mathbb{R}}$ is a representative in $\mathcal{LO}_{\omega_1}$ of some separable uncountable linear order.

Definition 6.22. Let $\leq_C \in \mathcal{LO}_{\omega_1}$ be such that $(\omega_1, \leq_C)$ is Countryman.[2]

Corollary 6.23. $\mathcal{B}_5 := \{\in \restriction (\omega_1)^2, \ni \restriction (\omega_1)^2, \leq_{\mathbb{R}}, \leq_C, \leq_C^*\}$ *is an antichain in* $\mathcal{LO}_{\omega_1}$.

The definition of $\leq_{\mathbb{R}}$ and $\leq_C$ seem somewhat odd, as the choice of representatives is completely arbitrary. However, we have the following interesting consistency results.

Fact 6.24 (Baumgartner, 1973)**.** *The proper forcing axiom (PFA) implies that* $\leq_{\mathbb{R}}$ *is co-initial in* $\{R \in \mathcal{LO}_{\omega_1} \mid (\omega_1, R) \textit{ is separable}\}$.

Fact 6.25 (Moore, 2006)**.** *The proper forcing axiom (PFA) implies that* $\mathcal{B}_5$ *is co-initial in* $\mathcal{LO}_{\omega_1}$.

For completeness, let us also mention the following.

Fact 6.26 (Sierpiński, 1932)**.** *The continuum hypothesis implies that any co-initial set in* $\{R \in \mathcal{LO}_{\omega_1} \mid (\omega_1, R) \textit{ is separable}\}$ *has size* $2^{\aleph_1}$.

Our next task is to present a construction of a Countryman line. For this, we first recall the definition of a left lexicographic ordering.

Definition 6.27 (Left lexicographic ordering)**.** For any countable set w and any $R \in \mathcal{LO}_w$, we derive a total ordering $\sqsubseteq_R$ on ${}^{<\omega_1}w$, as follows. For $s, t \in {}^{<\omega_1}w$, we distinguish two cases:

- if s and t are compatible, then simply let $s \sqsubseteq_R t$ iff $s \subseteq t$;
- if s and t are incompatible, write $\Delta(s, t) := \min\{\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\alpha) \neq t(\alpha)\}$, and let $s \sqsubseteq_R t$ iff $(s(\Delta(s, t)), t(\Delta(s, t))) \in R$.

[2]We shall soon see it exists.

Exercise 6.28. *Prove that $\sqsubseteq_R$ is indeed a total ordering of ${}^{<\omega_1}w$.*

Definition 6.29. Let $\preceq_{\rho_1}$ be the element of $\mathcal{LO}_{\omega_1}$ which is inherited from $(\mathcal{T}(\rho_1), \sqsubseteq_\in)$ by identifying every ordinal $\alpha < \omega_1$ with the fiber $\rho_{1\alpha}$. That is, for $\alpha, \beta < \omega_1$, we let $\alpha \preceq_{\rho_1} \beta$ iff $\rho_{1\alpha} \subseteq \rho_{1\beta}$ or $\rho_{1\alpha}(\Delta(\alpha,\beta)) < \rho_{1\beta}(\Delta(\alpha,\beta))$.

Theorem 6.30. *$(\omega_1, \preceq_{\rho_1})$ is a Countryman line.*

Proof. We shall define a coloring $c : (\omega_1)^2 \to V_\omega$ such that the preimage of any singleton would make a $(\preceq_{\rho_1})^2$-chain.

For all $\delta < \omega_1$, let $c(\delta,\delta) = 0$.

Next, let $\gamma < \delta < \omega_1$ be arbitrary ordinals. Define:

- $\mathsf{d}(\gamma,\delta) := \{\tau < \gamma \mid \rho_{1\gamma}(\tau) \neq \rho_{1\delta}(\tau)\}$;
- $n_{\gamma,\delta} := \max\{0, \rho_{1\gamma}(\tau), \rho_{1\delta}(\tau) \mid \tau \in \mathsf{d}(\gamma,\delta)\}$;
- $E_{\gamma,\delta} := \{\tau \leq \gamma \mid \rho_{1\gamma}(\tau) \leq n_{\gamma,\delta} \text{ or } \rho_{1\delta}(\tau) \leq n_{\gamma,\delta}\}$.

Then $\mathsf{d}(\gamma,\delta) \subseteq E_{\gamma,\delta}$, and by our convention to let $\rho_1(\gamma,\gamma) = 0$ for all $\gamma < \omega_1$, we have $\max(E_{\gamma,\delta}) = \gamma$. As $|E_{\gamma,\delta}| < \omega$, let us define the following objects:

- $m_{\gamma,\delta} := |E_{\gamma,\delta}|$;
- $\pi_{\gamma,\delta} : m_{\gamma,\delta} \leftrightarrow E_{\gamma,\delta}$ be the order-preserving bijection;
- $f_{\gamma,\delta} : m_{\gamma,\delta} \to \omega$ be such that $f_{\gamma,\delta}(i) = \rho_{1\gamma}(\pi_{\gamma,\delta}(i))$ for all $i < m_{\gamma,\delta}$;
- $g_{\gamma,\delta} : m_{\gamma,\delta} \to \omega$ be such that $g_{\gamma,\delta}(i) = \rho_{1\delta}(\pi_{\gamma,\delta}(i))$ for all $i < m_{\gamma,\delta}$.

Finally, let:

- $c(\gamma,\delta) = \{(0,n_{\gamma,\delta}), (1,m_{\gamma,\delta}), (2,f_{\gamma,\delta}), (3,g_{\gamma,\delta})\}$, and
- $c(\delta,\gamma) = \{(0,n_{\gamma,\delta}), (1,m_{\gamma,\delta}), (2,f_{\gamma,\delta}), (3,g_{\gamma,\delta}), (4,4)\}$.

Now, suppose that we are given ordinals $\alpha,\beta,\gamma,\delta < \omega_1$ such that $c(\alpha,\beta) = c(\gamma,\delta)$, say it is x. Note:

- If $\alpha = \gamma$, then $\{(\alpha,\beta),(\gamma,\delta)\} = \{(\alpha,\beta),(\alpha,\delta)\}$ is a chain.
- If $x = 0$, then $\{(\alpha,\beta),(\gamma,\delta)\} = \{(\beta,\beta),(\delta,\delta)\}$ is a chain.
- If $(4,4) \in x$, then $c(\beta,\alpha) = c(\delta,\gamma) = x \setminus \{(4,4)\}$ and $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain iff $\{(\beta,\alpha),(\delta,\gamma)\}$ is a chain.

Consequently, it suffices to prove that $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain, under the assumption that $\alpha \neq \gamma$ and x is of the form

$$x = \{(0,n),(1,m+1),(2,f),(3,g)\}.$$

In particular, $\alpha < \beta$ and $\gamma < \delta$.

TO BE CONTINUED NEXT WEEK... □