Definition 7.1. An uncountable linear order $(C,\leq)$ is said to be a *Countryman line* iff $(C^2,\leq^2)$ may be covered by countably many chains.

Exercise 7.2. *If $(C,\leq)$ is a Countryman line, then for each integer $n\geq 2$, $(C^n,\leq^n)$ may be covered by countably many chains.*

For all $\alpha,\beta\in\omega_1$, let $\mathsf{d}(\alpha,\beta)=\{\tau<\min\{\alpha,\beta\}\mid \rho_{1\alpha}(\tau)\neq\rho_{1\beta}(\tau)\}$, and, if $\rho_{1\alpha}$ and $\rho_{1\beta}$ are incompatible, then let $\Delta(\alpha,\beta):=\min(\mathsf{d}(\alpha,\beta))$. Note that by coherence of ρ_1, $\mathsf{d}(\alpha,\beta)$ is finite for any choice of $\alpha,\beta<\omega_1$.

Definition 7.3. For all $\alpha,\beta<\omega_1$, let $\alpha\preceq_{\rho_1}\beta$ iff one the two hold:

- $\rho_{1\alpha}$ and $\rho_{1\beta}$ are compatible, and $\rho_{1\alpha}\subseteq\rho_{1\beta}$;
- $\rho_{1\alpha}$ and $\rho_{1\beta}$ are incompatible, and $\rho_{1\alpha}(\Delta(\alpha,\beta))<\rho_{1\beta}(\Delta(\alpha,\beta))$.

Theorem 7.4. $(\omega_1,\preceq_{\rho_1})$ *is a Countryman line.*

Proof. We shall define a coloring $c:(\omega_1)^2\rightarrow V_\omega$ such that the preimage of any singleton would make a $(\preceq_{\rho_1})^2$-chain.

For all $\delta<\omega_1$, let $c(\delta,\delta)=0$.

Next, let $\gamma<\delta<\omega_1$ be arbitrary ordinals. Define:

- $n_{\gamma,\delta}:=\max\{0,\rho_{1\gamma}(\tau),\rho_{1\delta}(\tau)\mid\tau\in\mathsf{d}(\gamma,\delta)\}$;
- $E_{\gamma,\delta}:=\{\tau\leq\gamma\mid\rho_{1\gamma}(\tau)\leq n_{\gamma,\delta}\text{ or }\rho_{1\delta}(\tau)\leq n_{\gamma,\delta}\}$.

By coherence of ρ_1, $n_{\gamma,\delta}$ is an integer, and since ρ_1 is finite-to-one, $E_{\gamma,\delta}$ is finite. Clearly, $\mathsf{d}(\gamma,\delta)\subseteq E_{\gamma,\delta}$, and by our convention to let $\rho_1(\gamma,\gamma)=0$ for all $\gamma<\omega_1$, we have $\max(E_{\gamma,\delta})=\gamma$. Let:

- $m_{\gamma,\delta}:=|E_{\gamma,\delta}|$;
- $\pi_{\gamma,\delta}:m_{\gamma,\delta}\leftrightarrow E_{\gamma,\delta}$ be the order-preserving bijection;
- $f_{\gamma,\delta}:m_{\gamma,\delta}\rightarrow\omega$ be $f_{\gamma,\delta}:=\rho_{1\gamma}\circ\pi_{\gamma,\delta}$;
- $g_{\gamma,\delta}:m_{\gamma,\delta}\rightarrow\omega$ be $g_{\gamma,\delta}:=\rho_{1\delta}\circ\pi_{\gamma,\delta}$.

Finally, let:

- $c(\gamma,\delta)=\{(0,n_{\gamma,\delta}),(1,m_{\gamma,\delta}),(2,f_{\gamma,\delta}),(3,g_{\gamma,\delta})\}$, and
- $c(\delta,\gamma)=\{(0,n_{\gamma,\delta}),(1,m_{\gamma,\delta}),(2,f_{\gamma,\delta}),(3,g_{\gamma,\delta}),(4,4)\}$.

Now, suppose that we are given ordinals $\alpha,\beta,\gamma,\delta<\omega_1$ such that $c(\alpha,\beta)=c(\gamma,\delta)$, say it is x. Note:

- If $\alpha=\gamma$, then $\{(\alpha,\beta),(\gamma,\delta)\}=\{(\alpha,\beta),(\alpha,\delta)\}$ is a chain.
- If $x=0$, then $\{(\alpha,\beta),(\gamma,\delta)\}=\{(\beta,\beta),(\delta,\delta)\}$ is a chain.
- If $(4,4)\in x$, then $c(\beta,\alpha)=c(\delta,\gamma)=x\setminus\{(4,4)\}$ and $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain iff $\{(\beta,\alpha),(\delta,\gamma)\}$ is a chain.

Consequently, it suffices to prove that $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain, under the assumption that $\alpha\neq\gamma$ and x is of the form

$$x=\{(0,n),(1,m+1),(2,f),(3,g)\}.$$

In particular, $\alpha<\beta$ and $\gamma<\delta$.

Claim 7.4.1. *(1) $\mathsf{d}(\alpha,\gamma)\neq\emptyset$ and $\mathsf{d}(\beta,\delta)\neq\emptyset$;*
(2) $\Delta(\alpha,\gamma)\notin E_{\alpha,\beta}\cap E_{\gamma,\delta}$ and $\Delta(\beta,\delta)\notin E_{\alpha,\beta}\cap E_{\gamma,\delta}$.

Proof. By $\pi_{\alpha,\beta}(m)=\alpha\neq\gamma=\pi_{\gamma,\delta}(m)$, let $i\leq m$ be the least such that $\pi_{\alpha,\beta}(i)\neq\pi_{\gamma,\delta}(i)$. Without loss of generality, $\pi_{\alpha,\beta}(i)<\pi_{\gamma,\delta}(i)$.[1]

By $\pi_{\alpha,\beta}(j)=\pi_{\gamma,\delta}(j)$ for all $j<i$ and $\pi_{\alpha,\beta}(i)<\pi_{\gamma,\delta}(i)$, we have $\pi_{\alpha,\beta}(i)\in E_{\alpha,\beta}\cap\gamma\setminus E_{\gamma,\delta}$, so that $\rho_{1\alpha}(\pi_{\alpha,\beta}(i))\leq n<\rho_{1\gamma}(\pi_{\alpha,\beta}(i))$ and $\rho_{1\beta}(\pi_{\alpha,\beta}(i))\leq n<\rho_{1\delta}(\pi_{\alpha,\beta}(i))$. In particular, $\pi_{\alpha,\beta}(i)\in\mathsf{d}(\alpha,\gamma)\cap\mathsf{d}(\beta,\delta)$, so that $\Delta(\alpha,\gamma)\leq\pi_{\alpha,\beta}(i)$ and $\Delta(\beta,\delta)\leq\pi_{\alpha,\beta}(i)$.

▶ If $\Delta(\alpha,\gamma)\in E_{\alpha,\beta}\cap E_{\gamma,\delta}$, then pick $j\leq i$ such that $\pi_{\alpha,\beta}(j)=\Delta(\alpha,\gamma)$. By $\pi_{\alpha,\beta}(i)\neq\pi_{\gamma,\delta}(i)$, it must be the case that $j<i$. But then, the minimality of i implies that $\pi_{\gamma,\delta}(i)=\Delta(\alpha,\gamma)$, so that $\rho_{1\alpha}(\Delta(\alpha,\gamma))=f(j)=\rho_{1\gamma}(\Delta(\alpha,\gamma))$. This is a contradiction.

▶ If $\Delta(\beta,\delta)\in E_{\alpha,\beta}\cap E_{\gamma,\delta}$, then pick $j\leq i$ such that $\pi_{\alpha,\beta}(j)=\Delta(\beta,\delta)$. By $\pi_{\alpha,\beta}(i)\neq\pi_{\gamma,\delta}(i)$, it must be the case that $j<i$. But then, the minimality of i implies that $\rho_{1\beta}(\Delta(\beta,\delta))=g(j)=\rho_{1\delta}(\Delta(\beta,\delta))$, which is a contradiction. □

Claim 7.4.2. $(E_{\alpha,\beta}\triangle E_{\gamma,\delta})\cap\alpha\cap\gamma\subseteq\mathsf{d}(\alpha,\gamma)\cap\mathsf{d}(\beta,\delta)$.

Proof. Let $\tau\in(E_{\alpha,\beta}\triangle E_{\gamma,\delta})\cap\alpha\cap\gamma$ be arbitrary.

▶ If $\tau\in E_{\alpha,\beta}\setminus E_{\gamma,\delta}$, then $\rho_{1\alpha}(\tau)\leq n<\rho_{1\gamma}(\tau)$ and $\rho_{1\beta}(\tau)\leq n<\rho_{1\delta}(\tau)$, so that $\tau\in\mathsf{d}(\alpha,\gamma)\cap\mathsf{d}(\beta,\delta)$.

▶ If $\tau\in E_{\gamma,\delta}\setminus E_{\alpha,\beta}$, then $\rho_{1\gamma}(\tau)\leq n<\rho_{1\alpha}(\tau)$ and $\rho_{1\delta}(\tau)\leq n<\rho_{1\beta}(\tau)$, so that $\tau\in\mathsf{d}(\alpha,\gamma)\cap\mathsf{d}(\beta,\delta)$. □

Claim 7.4.3. $\Delta(\alpha,\gamma)=\Delta(\beta,\delta)$.

Proof. We have $\Delta(\alpha,\gamma)<\alpha<\beta$ and $\Delta(\alpha,\gamma)<\gamma<\delta$, so that $\Delta(\alpha,\gamma)\in\alpha\cap\gamma$.

▶ If $\Delta(\alpha,\gamma)\in E_{\alpha,\beta}\triangle E_{\gamma,\delta}$, then by Claim 7.4.2, $\Delta(\alpha,\gamma)\in\mathsf{d}(\beta,\delta)$.

▶ If $\Delta(\alpha,\gamma)\notin E_{\alpha,\beta}\triangle E_{\gamma,\delta}$, then by Claim 7.4.1, $\Delta(\alpha,\gamma)\notin E_{\alpha,\beta}\cup E_{\gamma,\delta}$, so that $\rho_{1\beta}(\Delta(\alpha,\gamma))=\rho_{1\alpha}(\Delta(\alpha,\gamma))\neq\rho_{1\gamma}(\Delta(\alpha,\gamma))=\rho_{1\delta}(\Delta(\alpha,\gamma))$ and hence again $\Delta(\alpha,\gamma)\in\mathsf{d}(\beta,\delta)$.

Consequently, $\Delta(\beta,\delta)\leq\Delta(\alpha,\gamma)$. In particular, $\Delta(\beta,\delta)\in\alpha\cap\gamma$.

▶ If $\Delta(\beta,\delta)\in E_{\alpha,\beta}\triangle E_{\gamma,\delta}$, then by Claim 7.4.2, $\Delta(\beta,\delta)\in\mathsf{d}(\alpha,\gamma)$.

▶ If $\Delta(\beta,\delta)\notin E_{\alpha,\beta}\triangle E_{\gamma,\delta}$, then by Claim 7.4.1, $\Delta(\beta,\delta)\notin E_{\alpha,\beta}\cup E_{\gamma,\delta}$, so that $\rho_{1\alpha}(\Delta(\beta,\delta))=\rho_{1\beta}(\Delta(\beta,\delta))\neq\rho_{1\delta}(\Delta(\beta,\delta))=\rho_{1\gamma}(\Delta(\beta,\delta))$ and hence again $\Delta(\beta,\delta)\in\mathsf{d}(\alpha,\gamma)$.

Consequently, $\Delta(\alpha,\gamma)\leq\Delta(\beta,\delta)$, and we are done. □

Without loss of generality, suppose that $\alpha\preceq_{\rho_1}\gamma$, and let us show that $\beta\preceq_{\rho_1}\delta$.

[1]Note that all we know is that $\alpha<\beta$ and $\gamma<\delta$.

By Claims 7.4.1 and 7.4.3, the analysis splits into three cases:

▶ If $\Delta(\beta,\delta) \notin E_{\alpha,\beta} \cup E_{\gamma,\delta}$, then by $\alpha \preceq_{\rho_1} \gamma$:

$$\rho_{1\delta}(\Delta(\beta,\delta)) = \rho_{1\gamma}(\Delta(\alpha,\gamma)) > \rho_{1\alpha}(\Delta(\alpha,\gamma)) = \rho_{1\beta}(\Delta(\beta,\delta)),$$

so that $\beta \preceq_{\rho_1} \delta$, as sought.

▶ If $\Delta(\beta,\delta) \in E_{\alpha,\beta} \setminus E_{\gamma,\delta}$, then

$$\rho_{1\beta}(\Delta(\beta,\delta)) \leq n < \rho_{1\delta}(\Delta(\beta,\delta)),$$

so that $\beta \preceq_{\rho_1} \delta$, as sought.

▶ If $\Delta(\alpha,\gamma) \in E_{\gamma,\delta} \setminus E_{\alpha,\beta}$, then

$$\rho_{1\gamma}(\Delta(\alpha,\gamma)) \leq n < \rho_{1\alpha}(\Delta(\alpha,\gamma)),$$

contradicting the fact that $\alpha \preceq_{\rho_1} \gamma$. So this case is void. □

The preceding proof only used the facts that ρ_1 is finite-to-one and coherent. An elaboration of this proof yields the following generalization.

Exercise 7.5. *If $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is an $\mathbb{R}$-embeddable coherent Aronszajn tree, then $(\mathcal{T}, \leq_{\in})$ is Countryman.*

Exercise 7.6 (Peng, 2013)**.** *Prove that $(\mathcal{T}(\rho_2), \leq_{\in})$ and $(\mathcal{T}_\pi(\rho_2), \leq_{<})$ are isomorphic.*

Conclude that $(\omega_1, \preceq_{\rho_2})$ is Countryman.

So, we already have two examples of Countryman lines: one that is read from the special tree $\mathcal{T}(\rho_2)$, and one that is a read from a (consistently, nonspecial) Aronszajn tree $\mathcal{T}(\rho_1)$. But why does it have to be Aronszajn in the first place?

Definition 7.7. A poset $(T, \leq_T)$ is a *tree* if for all $x \in T$, the set $x_\downarrow := \{y \in P \mid y <_T x\}$ is well-ordered by $\leq_T$. The α^{th}-level of the tree, denoted T_α consists of all $x \in P$ such that $\mathrm{otp}(x_\downarrow, <_T) = \alpha$.

A tree $(T, \leq_T)$ is *Aronszajn* if $0 < |T_\alpha| \leq \aleph_0$ for all $\alpha < \omega_1$ and there are no uncountable chains.

Definition 7.8. For a linearly ordered set $(L, \leq)$, let $\mathrm{Conv}(L, \leq)$ denote the collection of all convex subsets of $(L, \leq)$, i.e., the collection of all $U \subseteq L$ with the property that for every $x \leq y \leq z$ in L: if $x, z \in U$, then $y \in U$.

Given a function $f : \mathrm{Conv}(L, \leq) \to L$ satisfying $f(U) \in U \setminus \{\min(U)\}$ for all U with $|U| \geq 2$, we derive $\hat{f} : \mathrm{Conv}(L, \leq) \to \mathcal{P}(\mathrm{Conv}(L, \leq))$ by stipulating

$$\hat{f}(U) := \begin{cases} \{\{y \in U \mid y < f(U)\}, \{y \in U \mid f(U) \leq y\}\}, & \text{if } |U| \geq 2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 7.9 (The f-partition tree)**.** Given a linear order $(L, \leq)$ and a function f as above, one defines $\mathcal{T}^f \subseteq \mathrm{Conv}(L, \leq)$ so that $(\mathcal{T}^f, \supseteq)$ would make a tree-ordering.

This is done by recursion on $\alpha \in \mathrm{Ord}$:

- $T^f_0 := \{L\}$;
- $T^f_{\alpha+1} := \bigcup\{\hat{f}(U) \mid U \in T^f_\alpha\}$;
- $T^f_\alpha := \{\bigcap \mathrm{range}(b) \mid b \in \prod_{\beta<\alpha} T^f_\beta, (\mathrm{range}(b), \supseteq) \text{ is a chain}\} \setminus \{\emptyset\}$ for limit nonzero α.

Finally, put $\mathcal{T}^f := \bigcup\{T^f_\alpha \mid \alpha \in \mathrm{Ord}\}$.

Notice that for every chain $\mathcal{B}$ in $(T^f, \supseteq)$, if $|\bigcap \mathcal{B}| \geq 2$, then $\bigcap \mathcal{B}$ is a non-terminal node in T^f.

Exercise 7.10. *Prove that for any ordinal α, T^f_α consists of pairwise disjoint sets, and is equal to the α^{th} level of $(\mathcal{T}^f, \supseteq)$.*

Exercise 7.11. *Prove that $T^f_\alpha \cap T^f_\beta = \emptyset$ whenever $\alpha \neq \beta$.*

It follows that $T^f_\alpha = \emptyset$ whenever $\alpha \geq (2^{|L|})^+$. The *height* of $\mathcal{T}^f$ is thus defined to be the least α for which $T^f_\alpha = \emptyset$.

Lemma 7.12. *If $(C, \leq)$ is a Countryman line, then for every choice of f, $(\mathcal{T}^f, \supseteq)$ is an Aronszajn tree.*

Proof. Our proof will only make use of the fact that $(C, \leq)$ does not embed uncountable separable linear orders, nor $(\omega_1, \in)$ nor $(\omega_1, \ni)$.

Claim 7.12.1. *$|T^f_\alpha| \leq \aleph_0$ for all $\alpha < \omega_1$.*

Proof. Suppose not, and let $\alpha < \omega_1$ be the least for which $|T^f_\alpha| > \aleph_0$. For each $U \in T_\alpha$, pick $x_U \in U$. Now, if U, V are two distinct elements of T_α, then $x_U \neq x_V$, say $x_U < x_V$, and then by definition of the tree, the set $\{W \in \bigcup_{\beta<\alpha} T^f_\beta \mid \{x_U, x_V\} \subseteq W\}$ contains a maximal element with respect to the $\supseteq$-ordering. Let W be this maximal element. Then $x_U \leq f(W) \leq x_V$.

Thus, the countable set $\{f(W) \mid W \in \bigcup_{\beta<\alpha} T^f_\beta\}$ witnesses that $\{x_U \mid U \in T_\alpha\}$ is an uncountable separable suborder of $(C, \leq)$. This is a contradiction. □

Claim 7.12.2. *$T^f_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.*

Proof. Suppose not, so that T^f has a countable height, say δ. Then by the previous claim $D := \{f(U) \mid U \in T^f\}$ is countable. We shall reach a contradiction by showing that D witnesses that $(C, \leq)$ is separable.

Let $x < z$ be arbitrary elements of C. Let $\alpha \leq \delta$ be the least ordinal such that "$\{x, z\} \subseteq U$" does not hold for any $U \in T^f_\alpha$. Clearly, $\alpha > 0$.

For all $\beta < \alpha$, there exists $U \in T^f_\beta$ such that $\{x, z\} \subseteq U$. So α must be a successor ordinal. Let $U \in T^f_{\alpha-1}$ be such that $\{x, z\} \subseteq U$.

▶ If $z < f(U)$, then $\{x, z\} \subseteq \{y \in U \mid y < f(U)\} \in T^f_\alpha$, contradicting the choice of α.

▶ If $x \leq f(U) \leq z$, then by $f(U) \in D$, we are good.

▶ If $f(U) < x$, then $\{x, z\} \subseteq \{x \in U \mid x \geq f(U)\} \in T^f_\alpha$, contradicting the choice of α. □

Claim 7.12.3. *T^f does not have an uncountable branch. In particular, $T^f_{\omega_1}$ is empty.*

Proof. Suppose not. Pick $\langle U_\alpha \mid \alpha < \omega_1\rangle \in \prod_{\alpha<\omega_1} T^f_\alpha$ such that $\{U_\alpha \mid \alpha < \omega_1\}$ is linearly ordered by $\supseteq$. In particular, $|U_\alpha| \geq 2$ for all $\alpha < \omega_1$.

Define $g : \omega_1 \to 2$ by stipulating

$$g(\alpha) := \begin{cases} 1, & \text{if } U_{\alpha+1} = \{y \in U_\alpha \mid y \geq f(U_\alpha)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Pick an uncountable $Y \subseteq \omega_1$ on which g is constant.

▶ If $g[Y] = \{1\}$, then for all $\alpha < \beta$ both from Y, we have

$$f(U_\beta) \in U_\beta \subseteq U_{\alpha+1} = \{y \in U_\alpha \mid y \geq f(U_\alpha)\},$$

so that $f(U_\beta) \geq f(U_\alpha)$. Since $f(U_\alpha)$ is the minimal element of $U_{\alpha+1} \supseteq U_\beta$, and $f(U_\beta) \neq \min(U_\beta)$, we moreover get that $f(U_\beta) > f(U_\alpha)$. Consequently, $\{f(U_\alpha) \mid \alpha \in Y\}$ is a copy of $(\omega_1, \in)$ in $(C, \leq)$, which is a contradiction.

▶ If $g[Y] = \{0\}$, then for all $\alpha < \beta$ both from Y, we have

$$f(U_\beta) \in U_\beta \subseteq U_{\alpha+1} = \{y \in U_\alpha \mid y < f(U_\alpha)\},$$

and hence $f(U_\beta) < f(U_\alpha)$. Consequently, $\{f(U_\alpha) \mid \alpha \in Y\}$ is a copy of $(\omega_1, \ni)$ in $(C, \leq)$, which is again a contradiction. □

This completes the proof. □