Families of pairwise disjoint sets play a special rule in combinatorics. Unfortunately, often, such families are unavailable, and so, one is lead to settle for a slightly less nice families, called, Δ-*systems.* A Δ-system is a family of sets $\mathcal{A}$ for which there exist some auxiliary set r (the *root*) such that $a \cap b = r$ for all two distinct $a, b \in \mathcal{A}$. So a family of pairwise disjoint sets is the special case of a Δ-system with an empty root.

Lemma 8.1 (Δ-system lemma)**.** *Suppose that κ is a regular uncountable cardinal, $A \in [\kappa]^\kappa$, and $\{a_\alpha \mid \alpha \in A\}$ is an injective enumeration of finite subsets of κ.*

Then there exist $B \in [A]^\kappa$, $r \in [\kappa]^{<\omega}$, and $n < \omega$ such that:

- *$|a_\alpha| = n$ for all $\alpha \in B$;*
- *$a_\alpha \cap a_\beta = r$ for all $\alpha < \beta$ both from B;*
- *$\sup(r) < \alpha < \min(a_\delta \setminus r) \leq \max(a_\delta \setminus r) < \beta$ for all $\alpha < \delta < \beta$ in B.*

Proof. Since κ is regular and uncountable, there must exist some $n < \omega$ for which $\{\alpha \in A \mid |a_\alpha| = n\}$ has size κ. Thus, without loss of generality, we assume that $\alpha \mapsto |a_\alpha|$ is constant over A, with some value n. We now prove the lemma by induction on this n.

► For $n = 1$, we let $r := \emptyset$ and recursively determine a κ-sized subset $B \subseteq A \setminus \{0\}$ such that $\alpha < \min(a_\delta) = \max(a_\delta) < \beta$ for all $\alpha < \delta < \beta$ in B.

► Next, suppose that the claim holds for n, and let us prove it for $n + 1$.

►► If $\{\min(a_\alpha) \mid \alpha < \omega_1\}$ has size κ, then we let $r := \emptyset$, and recursively determine a κ-sized subset $B \subseteq A \setminus \{0\}$ such that $\alpha < \min(a_\delta) < \max(a_\delta) < \beta$ for all $\alpha < \delta < \beta$ in B.

►► Otherwise, there exists some $\varepsilon < \kappa$ such that $\{\alpha \in A \mid \min(a_\alpha) = \varepsilon\}$ has size κ. Now, appeal to the induction hypothesis with $\{a_\alpha \setminus \{\varepsilon\} \mid \alpha \in A\}$ to find $B \in [A]^\kappa$ and $r \in [\kappa]^{<\omega}$, so that B and $r \cup \{\varepsilon\}$ are as sought. □

Definition 8.2. Elements x, y of a poset $(P, \leq)$ are said to be *compatible* if there exists some $z \in P$ such that $x \leq z$ and $y \leq z$. $A \subseteq P$ is a *strong antichain* if any two distinct elements x, y of A are not compatible.[1]

Definition 8.3. A poset $(P, \leq)$ is said to be *κ-cc* if it has no strong antichains of size κ.

[1] That is, for no $z \in P$ do we have $x \leq z$ and $y \leq z$.

A topological space (X,τ) is said to be κ-cc if the poset $(\tau\setminus\{\emptyset\},\supseteq)$ is κ-cc, that is, if any collection of pairwise disjoint open sets has size $<\kappa$. We know that the product of two separable spaces is again separable, and, more generally, that the product of two topological spaces of density κ is of density κ. What about the product of two κ-cc posets?

Exercise 8.4. *Prove that if there exist κ-cc posets $(P,\leq_P)$ and $(Q,\leq_Q)$ whose product is not κ-cc, then there exists a poset $(R,\leq)$ whose square $(R^2,\leq^2)$ is not κ-cc.*

Exercise 8.5 (Kurepa, 1952)**.** *If $(T,\leq)$ is a Souslin tree, then it is an example of an $\aleph_1$-cc poset whose square is not $\aleph_1$-cc.*

Lemma 8.6. *Suppose that κ is a regular cardinal, and Ramsey's theorem holds at the level of κ, that is, $\kappa\rightarrow(\kappa)^2_2$. Then for any κ-cc poset $(P,\leq)$, the square $(P^2,\leq^2)$ is again κ-cc.*

Proof. Towards a contradiction, suppose that $(P,\leq)$ is κ-cc, and $A\subseteq P^2$ is a strong antichain of size κ. For all distinct $(x,x'),(y,y')\in A$, if there exists $z,z'\in P$ such that $x,y\leq z$ and $x',y'\leq z'$, then (z,z') would contradict the fact that (x,x') and (y,y') are incompatible. It follows that we may define a symmetric coloring $c:[A]^2\rightarrow 2$ by letting $c((x,x'),(y,y'))=0$ iff x and y are incompatible.

Now, by Ramsey's theorem for κ, we may find some $H\subseteq\kappa$ of size κ which is c-homogeneous.

► If $c``[H]^2=\{0\}$, then $H_0=\{x\mid(x,x')\in H\}$ is a strong antichain in $(P,\leq)$, and hence has size $<\kappa$. Since $|H|=\kappa$ and the latter is regular, there must exist some $x\in H_0$ such that $H^x:=\{x'\mid(x,x')\in H\}$ has size κ. Since $(P,\leq)$ has the κ-cc, then we may find distinct $x',x''\in H^x$ that are compatible. Then (x,x') and (x,x'') are compatible elements of A, which is a contradiction.

► If $c``[H]^2=\{1\}$, then a similar argument would yield a contradiction. □

It follows that if we would like to cook up a κ-cc poset whose square is not κ-cc, then we have to assume that Ramsey's theorem fails at κ. But how badly must it fail? The next definition provides a measure of the extent to which Ramsey's theorem fails at κ.

Definition 8.7 (Shelah, 1988)**.** $\mathrm{Pr}_1(\kappa,\mu,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\mu$ such that for every family $\mathcal{A}\subseteq[\kappa]^{<\chi}$ of κ many pairwise disjoint sets, and every color $\gamma<\mu$, there exists some $a,b\in\mathcal{A}$ with $\sup(a)<\min(b)$ such that $c[a\times b]=\{\gamma\}$.

Notice that Ramsey's theorem is equivalent to the failure of $\mathrm{Pr}_1(\omega,2,2)$.

Exercise 8.8. *Prove that if κ is a strongly-compact cardinal, then $\Pr_1(\kappa, \mu, n)$ fails for any $\mu < \kappa$ and positive $n < \omega$.*

Lemma 8.9. $\Pr_1(\kappa, 2, 2)$ *holds for every singular cardinal κ.*

Proof. Given a singular cardinal κ, let $\langle \kappa_i \mid i < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of cardinals, converging to κ. Define $d : \kappa \to \mathrm{cf}(\kappa)$ by letting $d(\alpha) := \min\{i < \mathrm{cf}(\kappa) \mid \alpha \leq \kappa_i\}$ for all $\alpha < \kappa$. Then define $c : [\kappa]^2 \to 2$ by stipulating $c(\alpha, \beta) := 1$ iff $d(\alpha) = d(\beta)$.

Let $H \subseteq \kappa$ of size κ be arbitrary.

► Since $d[H] \subseteq \mathrm{cf}(\kappa) < \kappa = |H|$, we know that $d \restriction H$ is not injective, and so there exist $\alpha < \beta$ in H such that $c(\alpha, \beta) = 0$.

► Since $d^{-1}\{i\}$ has size $\leq \kappa_i < \kappa$ for all $i < \mathrm{cf}(\kappa)$, there must exist $\alpha' < \beta'$ in H such that $c(\alpha, \beta) = 1$. □

Exercise 8.10 (Erdős-Tarski, 1943 + Kurepa, 1963)**.** *If κ is a singular strong limit cardinal, then the square of any κ-cc poset is again κ-cc.*

Theorem 8.11 (Galvin, 1980)**.** *If κ is a regular cardinal, and $\Pr_1(\kappa, 2, \omega)$ holds, then there exists a κ-cc poset whose square is not κ-cc.*

Proof. Let $c : [\kappa]^2 \to 2$ be a witness to $\Pr_1(\kappa, 2, \omega)$. Let P be the set of all pairs (x, i) where:

- x is a finite subset of κ;
- $i < 2$;
- $c``[x]^2 \subseteq \{i\}$.

We let $(x, i) \leq (y, j)$ iff $x \subseteq y$ and $i = j$.

The following is clear.

Claim 8.11.1. *For all $(x, i), (y, j) \in P$, (x, i) and (y, j) are compatible iff $i = j$ and $c``[x \cup y]^2 \subseteq \{i\}$.* □

Claim 8.11.2. $(P^2, \leq^2)$ *is not κ-cc.*

Proof. For every $\alpha < \kappa$, we have $(\{\alpha\}, i) \in P$ both for $i = 0$ and for $i = 1$, so let $A := \{((\{\alpha\}, 0), (\{\alpha\}, 1)) \mid \alpha < \kappa\}$. Clearly, A is a strong antichain of size κ in $(P^2, \leq^2)$. □

Claim 8.11.3. $(P, \leq)$ *is κ-cc.*

Proof. Suppose that we are given $A \subseteq P$ of size κ. By Ramsey's theorem, κ must be uncountable. So, by shrinking A, we may assume the existence of $n < \omega$ and $j < 2$ such that $|x| = n$ and $i = j$ for all $(x, i) \in A$. Let $\mathcal{A} = \{x \mid (x, j) \in A\}$. Then $\mathcal{A}$ has size κ, and by the Δ-system lemma, let us pick some $\mathcal{B} \subseteq \mathcal{A}$ of size κ, and a set r such that $x \cap y = r$ for all two distinct $x, y \in \mathcal{B}$. Let $\mathcal{B}' = \{x \setminus r \mid x \in \mathcal{B}\}$. Then $\mathcal{B}$ is a family of κ many pairwise disjoint finite subsets of κ. By the choice of c, let us pick $a, b \in \mathcal{B}$ such that $\sup(a) < \min(b)$ and $c[a \times b] = \{j\}$. Let $x := a \cup r$ and $y := b \cup r$. Then $(x, j), (y, j) \in A$. We shall reach a contradiction by verifying that the two are compatible. Indeed, we shall show that $c``[x \cup y]^2 = \{j\}$.

Let $\alpha < \beta$ be arbitrary elements of $x \cup y$.

▶ If $\{\alpha, \beta\} \subseteq x$ or $\{\alpha, \beta\} \subseteq y$, then by $(x, j), (y, j) \in p$, we have $c(\alpha, \beta) = j$.

▶ If $\alpha \in x \setminus y$ and $\beta \in y \setminus x$, then $\alpha \in a$ and $\beta \in b$, and so by the choice of a and b, we have $c(\alpha, \beta) = j$.

▶ If $\alpha \in y \setminus x$ and $\beta \in x \setminus y$, then $\alpha \in b$ and $\beta \in a$, contradicting the fact that $\sup(a) < \min(b)$. □

This completes the proof. □

Exercise 8.12. *Prove that the forcing to add a Cohen real introduces* $\Pr_1(\omega_1, \omega_1, \omega)$.

Recall that a topological space $\mathbb{X} = (X, \tau)$ is said to be *Hausdorff* if for all two distinct points $x, y \in X$, there exist disjoint open sets U_x, U_y with $x \in U_x$ and $y \in U_y$. The space $\mathbb{X}$ is said to be *regular* if for every nonempty closed set F, and any point $x \notin F$, there exist disjoint open sets U_x, U_F with $x \in U_x$ and $F \subseteq U_F$. Equivalently, if for every point x and an open neighborhood U_x of x, there exists a closed subset $F \subseteq U_x$ where x is in its interior.

Definition 8.13 (Hajnal-Juhász, 1968)**.** An *L-space* is a regular Hausdorff topological space which is hereditarily Lindelöf but not separable.

An *S-space* is a regular Hausdorff topological space which is hereditarily separable but not Lindelöf.

A *strong L-space* (resp. *strong S-space*) is a topological space whose all finite powers are L-space (resp. S-space).

Exercise 8.14 (Zenor, 1980)**.** *There is a strong L-space iff there is a strong S-space.*

We now consider a stronger form of $\Pr_1$.

Definition 8.15 (Shelah, 1988)**.** $\Pr_0(\kappa, \mu, (\chi_0, \chi_1))$ asserts the existence of a coloring $c : [\kappa]^2 \to \mu$ such that for every:

- $\sigma_0 < \chi_0$, and a family $\mathcal{A}_0 \subseteq [\kappa]^{\sigma_0}$ of size κ, consisting of pairwise disjoint sets;
- $\sigma_1 < \chi_1$, and a family $\mathcal{A}_1 \subseteq [\kappa]^{\sigma_1}$ of size κ, consisting of pairwise disjoint sets;
- prescribed coloring $g : \sigma_0 \times \sigma_1 \to \mu$,

there exists some $(a_0, a_1) \in \mathcal{A}_0 \times \mathcal{A}_1$ with $\sup(a_0) < \min(a_1)$ such that

$$c(a_0(i), a_1(j)) = g(i,j) \text{ for all } (i,j) \in \sigma_0 \times \sigma_1.$$

Lemma 8.16 (Erdős). *$\mathrm{Pr}_1(\omega_1, \omega, 2)$ is equivalent to $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$.*

Proof. Let $c : [\omega_1]^2 \to \omega$ be a witness to $\mathrm{Pr}_1(\omega_1, \omega, 2)$. For every $\beta < \omega_1$, fix a surjection $\psi_\beta : \omega \to \beta$. Define $d : [\omega_1]^2 \to \omega_1$ by letting $d(\alpha, \beta) := \psi_\beta(c(\alpha, \beta))$ for all $\alpha < \beta < \omega_1$. Then d witnesses $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$. □

Exercise 8.17. *Prove that the following are equivalent:*

(1) $\mathrm{Pr}_1(\omega_1, \omega, \omega)$*;*
(2) $\mathrm{Pr}_1(\omega_1, \omega_1, \omega)$*;*
(3) $\mathrm{Pr}_0(\omega_1, \omega_1, (\omega, \omega))$.

Theorem 8.18 (Roitman?). *If $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$ holds, then there exists an L-space.*

Proof. Let $c : [\omega_1]^2 \to 2$ be a witness to $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$. We shall construct the L-space as a subspace of the product space ${}^{\omega_1}2$. The motivation comes from the following:

Claim 8.18.1. *Every subspace of ${}^{\omega_1}2$ is Hausdorff and regular.*

Proof. Recall that a basic open set in ${}^{\omega_1}2$ has the form $[s] = \{x \in {}^{\omega_1}2 \mid s \subseteq x\}$, where $s : a \to 2$ is a function from a finite subset of ω_1 to 2. Let $X \subseteq {}^{\omega_1}2$ be arbitrary.

▶ X is Hausdorff: if x, y are distinct points of X, then pick $\alpha < \omega_1$ such that $x(\alpha) \neq y(\alpha)$, and let $U_x := [\{(\alpha, x(\alpha)\}]$ and $U_y := [\{(\alpha, y(\alpha))\}]$. Then $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

▶ X is regular: if $x \in U_x$ and U_x is open, then by definition of the product topology, there exists some finite function s such that $x \in [s] \subseteq U_x$. For every $\alpha \in \mathrm{dom}(s)$, let $t_\alpha : \{\alpha\} \to 2$ be the unique function to satisfy $t_\alpha(\alpha) \neq s(\alpha)$. Then the complement of $[s]$ is equal to the finite union $\bigcup\{[t_\alpha] \mid \alpha \in \mathrm{dom}(s)\}$. This shows that $[s]$ is also closed. □

Let $X := \{x_\beta \mid \beta < \omega_1\}$, where for every $\beta < \omega_1$, $x_\beta : \omega_1 \to 2$ is defined by letting for all $\alpha < \omega_1$:

$$x_\beta(\alpha) := \begin{cases} c(\alpha, \beta), & \text{if } \alpha < \beta; \\ 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha > \beta. \end{cases}$$

Claim 8.18.2. *For every $\beta < \omega_1$, $\overline{\{x_\alpha \mid \alpha < \beta\}} \cap \{x_\gamma \mid \beta \leq \gamma < \omega_1\} = \emptyset$. In particular, X is not separable.*

Proof. Let $\beta < \omega_1$ be arbitrary. Let $\gamma \geq \beta$ be an arbitrary countable ordinal. We shall find an open neighborhood U of x_γ which is disjoint from $\{x_\alpha \mid \alpha < \beta\}$. Indeed, let $U := [s]$, where $s : \{\gamma\} \rightarrow \{1\}$ is a constant finite function. Then, $x_\gamma \in U$, while $\{x_\alpha \mid \alpha < \beta\} \cap U = \emptyset$. □

We remark that a space that can be well-ordered in such a way that every initial segment (according to this ordering) is closed — is said to be *left separated*.

TO BE CONTINUED NEXT WEEK. □