Recall that $\mathrm{Pr}_0(\kappa, \mu, (\chi_0, \chi_1))$ asserts the existence of a coloring $c : [\kappa]^2 \rightarrow \mu$ such that for every:

- $\sigma_0 < \chi_0$, and a family $\mathcal{A}_0 \subseteq [\kappa]^{\sigma_0}$ of size κ, consisting of pairwise disjoint sets;
- $\sigma_1 < \chi_1$, and a family $\mathcal{A}_1 \subseteq [\kappa]^{\sigma_1}$ of size κ, consisting of pairwise disjoint sets;
- prescribed coloring $g : \sigma_0 \times \sigma_1 \rightarrow \mu$,

there exists some $(a_0, a_1) \in \mathcal{A}_0 \times \mathcal{A}_1$ with $\sup(a_0) < \min(a_1)$ such that

$$c(a_0(i), a_1(j)) = g(i, j) \text{ for all } (i, j) \in \sigma_0 \times \sigma_1.$$

Theorem 9.1 (Roitman?). *If $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$ holds, then there exists an L-space.*

Proof. Let $c : [\omega_1]^2 \rightarrow 2$ be a witness to $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$. We shall construct the L-space as a subspace of the product space ${}^{\omega_1}2$, since every subspace of ${}^{\omega_1}2$ is Hausdorff and regular.[1]

Let $X := \{x_\beta \mid \beta < \omega_1\}$, where for every $\beta < \omega_1$, $x_\beta : \omega_1 \rightarrow 2$ is defined by letting for all $\alpha < \omega_1$:

$$x_\beta(\alpha) := \begin{cases} c(\alpha, \beta), & \text{if } \alpha < \beta; \\ 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha > \beta. \end{cases}$$

Last week, we showed that the natural ordering of X witnesses that X is left-separated. That is, for every $\beta < \omega_1$, $\overline{\{x_\alpha \mid \alpha < \beta\}} \cap \{x_\gamma \mid \beta \leq \gamma < \omega_1\} = \emptyset$. In particular, X is not separable.

Claim 9.1.1. *If X is not hereditarily Lindelöf, then X contains an uncountable discrete space.*

Proof. Suppose that Y is a subspace of X which is not Lindelöf. Fix an open cover $\mathcal{U}$ of Y that does not admit a countable subscover. Since the weight of ${}^{\omega_1}2$ is ω_1, we may assume that $\mathcal{U} = \{U_i \mid i < \omega_1\}$.

Define functions $f : \omega_1 \rightarrow \omega_1$ and $g : \omega_1 \rightarrow \omega_1$ by stipulating:

- $f(\alpha) := \min\{\beta < \omega_1 \mid x_\beta \in Y \setminus \bigcup\{U_i \mid i < \alpha\}\}$;
- $g(\alpha) := \min\{i < \omega_1 \mid x_{f(\alpha)} \in U_i\}$.

Since $\mathcal{U}$ is a cover, f is weakly increasing. As $g(\alpha) \geq \alpha$ for all $\alpha < \omega_1$, the range of f is uncountable. Consider the club $C := \{\gamma < \omega_1 \mid g[\gamma] \subseteq \gamma\}$. Then $f[C]$ is uncountable, and we may fix an uncountable $A \subseteq C$ such that $f \restriction A$ is one-to-one. Altogether, $f \restriction A$ is strictly increasing.

[1] Recall that a basic open subset of ${}^{\omega_1}2$ is $[s] := \{f : \omega_1 \rightarrow 2 \mid s \subseteq f\}$, where $s : a \rightarrow 2$ is a function, and $a \subseteq \omega_1$ is finite.

We claim that $D := \{x_{f(\alpha)} \mid \alpha \in A\}$ is discrete. To see this, fix an arbitrary $\alpha \in A$. Put

$$V_\alpha := U_{g(\alpha)} \setminus \overline{\{x_\beta \mid \beta < f(\alpha)\}}.$$

By Definition of $g(\alpha)$, $x_{f(\alpha)} \in U_{g(\alpha)}$. Since X is left separated as witnessed by the natural ordering, $x_{f(\alpha)} \notin \overline{\{x_\beta \mid \beta < f(\alpha)\}}$. So V_α is an open neighborhood of $x_{f(\alpha)}$. To see that $D \cap V_\alpha$ is a singleton, let $\gamma \in A$ be arbitrary. We consider two cases:

▶ If $\gamma < \alpha$, then $f(\gamma) < f(\alpha)$, and by definition of V_α, we have $x_{f(\gamma)} \notin V_\alpha$.

▶ If $\alpha < \gamma$, then $g(\alpha) < \gamma$ and so by definition of $f(\gamma)$, $x_{f(\gamma)} \notin U_{g(\alpha)}$, let alone $x_{f(\gamma)} \notin V_\alpha$. □

Thus, to complete the proof that X is an L-space, we suppose that $\{x_\beta \mid \beta \in B\}$ is discrete, for some fixed uncountable $B \subseteq \omega_1$, and derive a contradiction.

As $\{x_\beta \mid \beta \in B\}$ is discrete, there exists a collection of finite functions $\{s_\beta : a_\beta \to 2 \mid \beta \in B\}$ such that $\{\alpha \in B \mid x_\alpha \in [s_\beta]\} = \{\beta\}$ for all $\beta \in B$. By passing to an uncountable subset $A \subseteq B$ (via pigeonholes and the Δ-system lemma), we ensure the existence of $r \in [\omega_1]^{<\omega}$, $n < \omega$ and $f : n \to 2$ such that:

(1) $a_\alpha \cap a_\beta = r$ for all $\alpha < \beta$ in A;
(2) $\sup(r) < \alpha < \min(a_\delta \setminus r) \le \max(a_\delta \setminus r) < \beta$ for all $\alpha < \delta < \beta$ in A;
(3) $|a_\alpha| = n$;
(4) $s_\alpha(a_\alpha(i)) = f(i)$ for all $i < n$ and $\alpha \in A$.

Put $\mathcal{A} := \{a_\alpha \setminus r \mid \alpha \in A\}$, so that $\mathcal{A}$ is an uncountable family of pairwise disjoint finite subsets of ω_1. Define $g : (n - |r|) \times 1 \to 2$ by stipulating:

$$g(i, 0) := f(|r| + i).$$

As $c : [\omega_1]^2 \to 2$ witnesses $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$, let us pick $a \in \mathcal{A}$ and $\beta \in A$ with $\max(a) < \beta$ such that

$$c(a(j), \beta) = g(j, 0) \text{ for all } j < |a|.$$

Let $\alpha \in A$ be such that $a = a_\alpha \setminus r$. Then $\max(a_\alpha) < \beta$ and for all $j < n - |r|$, we have:

$$x_\beta(a_\alpha(|r|+j)) = c(a_\alpha(|r|+j), \beta) = g(j, 0) = f(|r|+j) = s_\alpha(a_\alpha(|r|+j)).$$

In other words, we have:

$$x_\beta(a_\alpha(i)) = s_\alpha(a_\alpha(i)), \quad (|r| \le i < n).$$

In parallel, by $\alpha, \beta \in A$, we get from Clause (4) above that for all $i < |r|$:

$$s_\alpha(a_\alpha(i)) = f(i) = s_\beta(a_\beta(i)).$$

By Clauses (1) and (2) above, we infer that $a_\alpha(i) = r(i) = a_\beta(i)$ for all $i < |r|$.

As $x_\beta \in [s_\beta]$, it must be the case that $x_\beta(a_\beta(i)) = s_\beta(a_\beta(i))$ for all $i < n$, and hence

$$x_\beta(a_\alpha(i)) = s_\alpha(a_\alpha(i)), \quad (i < |r|).$$

Altogether, we have established that $x_\beta \restriction a_\alpha = s_\alpha$. So $x_\beta \in [s_\alpha]$ contradicting the fact that $\alpha \neq \beta$. □

Unlike Tychonoff's theorem that asserts that the product of compact spaces is compact, there are Lindelöf spaces whose square is not Lindelöf. In fact:

Exercise 9.2 (Kunen, 1977). *Martin's Axiom,* $\mathrm{MA}_{\aleph_1}$*, implies that for any L-space (resp. S-space) X there exists some integer* $n \geq 2$ *such that* X^n *is not an L-space (resp. S-space).*

And yet, stronger failures of Ramsey's theorem gives us more:

Exercise 9.3. *Let* $n < \omega$ *be arbitrary.*

(1) *If* $\mathrm{Pr}_0(\omega_1, 2, (\omega, n+2))$ *holds, then there exists an L-space X such that* X^{n+1} *is still an L-space.*
(2) *If* $\mathrm{Pr}_0(\omega_1, 2, (n+2, \omega))$ *holds, then there exists an S-space X such that* X^{n+1} *is still an S-space.*

A useful hypothesis in this vein is the continuum hypothesis (CH):

Theorem 9.4 (Galvin, 1980). *CH implies* $\mathrm{Pr}_1(\omega_1, \omega_1, \omega)$.

Proof. By an exercise from last week, it suffices to get $\mathrm{Pr}_1(\omega_1, \omega, \omega)$. Consider the set F of all functions $f : \omega \to [\omega_1]^{<\omega}$. We have $|F| = |[\omega_1]^{<\omega}|^{\aleph_0} = (\aleph_1)^{\aleph_0}$. So, by the continuum hypothesis, $|F| = \aleph_1$. Let $\{f_\alpha \mid \alpha < \omega_1\}$ be some enumeration of F, such that for every $f \in F$, we have that $\{\alpha < \omega_1 \mid f_\alpha = f\}$ is uncountable.[2] Let us say that α is *good*, if $\langle f_\alpha(n) \mid n < \omega\rangle$ is a sequence of pairwise disjoint (finite) subsets of α.

The coloring c will be obtained as a limit of an increasing and continuous chain $\{c_\delta : [\delta]^2 \to \omega \mid \delta < \omega_1\}$, where for all $\delta < \omega_1$, we would have:

[2]For instance, let $\{h_\alpha \mid \alpha < \omega_1\}$ be an injective enumeration of F. Let $\pi : \omega_1 \leftrightarrow \omega_1 \times \omega_1$ be some bijection. Then set $f_\alpha = h_\beta$ whenever $\pi(\alpha) = (\beta, \gamma)$.

($*$) For every good $\alpha \leq \delta$, every $b \in [\delta \setminus \alpha]^{<\omega}$, and every color $\gamma < \omega$, the set

$$\mathcal{G}_\delta(\alpha, b, \gamma) := \{n < \omega \mid b = \emptyset \text{ or } c_\delta[f_\alpha(n) \times b] = \{\gamma\}\} \text{ is infinite.}$$

Let c_0 be the empty coloring. Next, suppose that $c_\delta : [\delta]^2 \to \omega$ has already been defined and satisfies $(*_\delta)$. We shall now define $c_{\delta+1}$. Let

$$\mathcal{A} := \{(\alpha, b, \gamma) \mid \alpha < \delta + 1 \text{ is good }, b \in [\delta + 1 \setminus \alpha]^{<\omega}, \gamma < \omega\}.$$

By $(*_\delta)$, we know that $\mathcal{G}_\delta(\alpha, b \cap \delta, \gamma)$ is infinite for all $(\alpha, b, \gamma) \in \mathcal{A}$.

Let $\{(\alpha_m, b_m, \gamma_m) \mid m < \omega\}$ be some enumeration of $\mathcal{A}$ in such a way that every element is enumerated infinitely often. We shall define an increasing sequence $\langle n_m \mid m < \omega\rangle \in \prod_{m<\omega} \mathcal{G}_\delta(\alpha_m, b_m \cap \delta, \gamma_m)$ by recursion, in such a way that $\langle f_{\alpha_m}(n_m) \mid m < \omega\rangle$ would consist of pairwise disjoint sets.

▶ Since $\mathcal{G}_\delta(\alpha_0, b_0 \cap \delta, \gamma_0)$ is infinite. Let n_0 denote its minimal element.

▶ Suppose that $m < \omega$ and $\langle n_k \mid k \leq m\rangle$ has already been defined. Since

$$\langle f_{\alpha_{m+1}}(n) \mid n \in \mathcal{G}_\delta(\alpha_{m+1}, b_{m+1} \cap \delta, \gamma_{m+1})\rangle$$

is an infinite sequence of pairwise disjoint sets, there are infinitely many $n \in \mathcal{G}_\delta(\alpha_{m+1}, b_{m+1} \cap \delta, \gamma_{m+1})$ such that $f_{\alpha_{m+1}}(n)$ is disjoint from the finite set $\bigcup_{k\leq m} f_{\alpha_k}(n_k)$. Thus, we fix $n_{m+1} \in \mathcal{G}_\delta(\alpha_{m+1}, b_{m+1} \cap \delta, \gamma_{m+1})$ in such a way that $f_{\alpha_{m+1}}(n_{m+1})$ is disjoint from $\bigcup_{k\leq m} f_{\alpha_k}(n_k)$, and $n_{m+1} > \max\{n_k \mid k \leq m\}$.

Finally, define $c_{\delta+1} \supseteq c_\delta$ in such a way that for all $\varepsilon < \delta$:

$$c_{\delta+1}(\varepsilon, \delta) := \begin{cases} 0, & \text{if } \varepsilon \notin \bigcup_{m<\omega} f_{\alpha_m}(n_m); \\ \gamma_m, & \text{if } \varepsilon \in f_{\alpha_m}(n_m). \end{cases}$$

Claim 9.4.1. $(*_{\delta+1})$ *holds.*

Proof. Fix a good $\alpha < \delta + 1$, $b \in [\delta + 1 \setminus \alpha]^{<\omega}$, and $\gamma < \omega$. Let $M := \{m < \omega \mid (\alpha_m, b_m, \gamma_m) = (\alpha, b, \gamma)\}$. We shall show that

$$\mathcal{G}_{\delta+1}(\alpha, b, \gamma) = \{n < \omega \mid b = \emptyset \text{ or } c_{\delta+1}[f_\alpha(n) \times b] = \{\gamma\}\}$$

contains the infinite set $\{n_m \mid m \in M\}$.

Fix $m \in M$, and let us show that $c_{\delta+1}[f_{\alpha_m}(n_m) \times b] = \{\gamma\}$. Pick $(\varepsilon, \beta) \in f_\alpha(n_m) \times b$.

▶ If $\beta < \delta$, then $(\varepsilon, \beta) \in f_{\alpha_m}(n_m) \times (b \cap \delta)$. Since $b \cap \delta \neq \emptyset$ and $n_m \in \mathcal{G}_\delta(\alpha, b \cap \delta, \gamma)$, we have $c_{\delta+1}(\varepsilon, \beta) = c_\delta(\varepsilon, \beta) = \gamma$.

▶ If $\beta = \delta$, then by $\varepsilon \in f_{\alpha_m}(n_m)$, we have $c_{\delta+1}(\varepsilon, \delta) = \gamma_m = \gamma$. □

Let $c := \bigcup_{\delta<\omega_1} c_\delta$.

Claim 9.4.2. *$c : [\omega_1]^2 \to \omega$ witnesses* $\mathrm{Pr}_1(\omega_1, \omega, \omega)$.

Proof. Suppose that $\mathcal{A}$ is uncountable subfamily of $[\omega_1]^{<\omega}$ consisting of pairwise disjoint sets, and $\gamma < \omega$. Fix some injection $f : \omega \to \mathcal{A}$. Then $f \in F$. Since $\bigcup \mathrm{Im}(f)$ is a countable set, there exists a large enough $\alpha < \omega_1$ such that $f_\alpha = f$ and $\bigcup \mathrm{Im}(f) \subseteq \alpha$. In particular, α is good. Pick some $b \in \mathcal{A}$ such that $\min(b) > \alpha$. Let $\delta := \max(b) + 1$. As α is good and smaller than δ, $b \in [\delta \setminus \alpha]^{<\omega}$ and $\gamma < \omega$. So by $(*_\delta)$, we may pick some n from the set

$$\mathcal{G}_\delta(\alpha, b, \gamma) := \{n < \omega \mid b = \emptyset \text{ or } c_\delta[f_\alpha(n) \times b] = \{\gamma\}\}.$$

Then $a := f_\alpha(n)$ is in $\mathcal{A}$, $\sup(a) < \alpha < \min(b)$ and $c[a \times b] = c_\delta[a \times b] = \{\gamma\}$. □

This completes the proof. □

The above proof shows a little more: CH entails a function $c : [\omega_1]^2 \to \omega_1$ such that for every $\gamma < \omega_1$, every countable family of pairwise disjoint subsets of ω_1, $\mathcal{A}$, and every uncountable family of pairwise disjoint subset of ω_1, $\mathcal{B}$, there exist $a \in \mathcal{A}$ and $b \in \mathcal{B}$ with $\sup(a) < \min(b)$ such that $c[a \times b] = \{\gamma\}$.

We remark that Galvin's proof of Theorem 9.4 above is a generalization of an older argument for getting a weaker coloring from CH:

Corollary 9.5 (Erdős-Hajnal-Rado, 1965)**.** *The continuum hypothesis implies* $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$.

Definition 9.6. A subset A of a poset $(P, \leq)$ is said to be *directed* if for every $x, y \in A$, there exists $z \in P$ such that $x \leq z$ and $y \leq z$.

$(P, \leq)$ is said to be *Knaster* if every uncountable $A \subseteq P$ contains an uncountable $B \subseteq A$ which is directed.

Exercise 9.7. *Prove that Martin's axiom implies that the following are equivalent:*

- *Continuum Hypothesis;*
- $\mathrm{Pr}_1(\omega_1, \omega_1, \omega)$*;*
- $\mathrm{Pr}_1(\omega_1, 2, \omega)$*;*
- *there exists an $\aleph_1$-cc poset whose square is not $\aleph_1$-cc;*
- *there exists an $\aleph_1$-cc poset which is not Knaster;*
- *there exists a strong L-space;*
- *there exists a strong S-space.*

Problem 9.8. *Is* $\mathrm{MA}_{\aleph_1}$ *equivalent to the productivity of the $\aleph_1$-cc?*

Now, we come back to ZFC.

Lemma 9.9. *Suppose that there exists a function $f : [\omega_1]^2 \to \omega_1$ with the property that for every uncountable $Z \subseteq \omega_1$, $f``[Z]^2$ covers a club. Then $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$ holds.*

Proof. Let $\langle S_\gamma \mid \gamma < \omega_1 \rangle$ be a partition of ω_1 into mutually disjoint stationary sets. Define $\psi : \omega_1 \to \omega_1$ by stipulating $\psi(\delta) = \gamma$ iff $\delta \in S_\gamma$. Define $c : [\omega_1]^2 \to \omega_1$ by letting $c := \psi \circ f$. To see that c witnesses $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$, fix an arbitrary uncountable set $Z \subseteq \omega_1$ along with an arbitrary color $\gamma < \omega_1$. Fix a club $C \subseteq \omega_1$ such that $f``[Z]^2 \supseteq C$.

Since S_γ is stationary, we may fix some $\delta \in C \cap S_\gamma$. Fix a pair of ordinals $\alpha < \beta$ from Z such that $f(\alpha, \beta) = \delta$. Then $c(\alpha, \beta) = \gamma$. $\square$

Next week, we shall use the preceding lemma to prove that $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$ holds. We shall also make use of the following observation.

Lemma 9.10. *For every $B \subseteq \omega_1$, the following set is a club:*

$$C^B := \{\delta < \omega_1 \mid B \subseteq \delta\} \cup \{\delta < \omega_1 \mid |B| = \aleph_1 \ \&\ \delta = \sup(B \cap \delta)\}.$$

Proof. If $|B| < \aleph_1$, then C^B is a final segment of ω_1, and hence a club. Otherwise, define $f : \omega_1 \to \omega_1$ by stipulating:

$$f(\alpha) := \min(B \setminus (\alpha + 1)).$$

Then C^B coincides with the club $\{\delta < \omega_1 \mid f[\delta] \subseteq \delta\}$. $\square$