Theorem 10.1 (Todorcevic, 1987)**.** $\mathrm{Pr}_1(\omega_1,\omega_1,2)$ *holds.*

Proof. Last week, we have noticed that to prove $\mathrm{Pr}_1(\omega_1,\omega_1,2)$, it suffices to exhibit a function $f:[\omega_1]^2\rightarrow\omega_1$ with the property that for every uncountable $Z\subseteq\omega_1$, $f``[Z]^2$ covers a club.

Fix ω_1 many distinct functions $\{r_\alpha\mid\alpha<\omega_1\}\subseteq{}^\omega 2$. For $\alpha\neq\beta$, let

$$\Delta(\alpha,\beta):=\min\{n<\omega\mid r_\alpha(n)\neq r_\beta(n)\}.$$

For each $\beta<\omega_1$, by $|\beta|\leq\aleph_0$, let us fix some injection $e_\beta:\beta\rightarrow\omega$. Then, for all $\alpha<\beta<\omega_1$, let

$$\Gamma(\alpha,\beta):=\{\gamma<\beta\mid e_\beta(\gamma)\geq\Delta(\alpha,\beta)\}\setminus\alpha,$$

so that $\Gamma(\alpha,\beta)\subseteq[\alpha,\beta)$. Finally, define $f:[\omega_1]^2\rightarrow\omega_1$ by letting for all $\alpha<\beta<\omega_1$:

$$f(\alpha,\beta):=\begin{cases}\min(\Gamma(\alpha,\beta)), & \text{if } \Gamma(\alpha,\beta)\neq\emptyset;\\ \alpha, & \text{otherwise.}\end{cases}$$

We shall want to show that f is as sought.

For every $Z\subseteq\omega_1$ and $g\in{}^{<\omega}2$, denote $B^Z_g:=\{\alpha\in Z\mid g\subseteq r_\alpha\}$.

Claim 10.1.1. *For any $Z\subseteq\omega_1$, the following set is a club:*

$$C_Z:=\left\{\delta<\omega_1\mid\forall g\in{}^{<\omega}2\left[(B^Z_g\subseteq\delta)\text{ or }(|B^Z_g|=\aleph_1\text{ and }\delta=\sup(B^Z_g\cap\delta))\right]\right\}.$$

Proof. Last week, we have noticed that for every $B\subseteq Z$,

$$C^B:=\{\delta<\omega_1\mid B\subseteq\delta\}\cup\{\delta<\omega_1\mid |B|=\aleph_1\text{ and }\delta=\sup(B\cap\delta)\}$$

is a club. In particular, $C_Z=\bigcap_{g\in{}^{<\omega}2}C^{B^Z_g}$ is the countable intersection of clubs. □

Thus, to complete our proof it suffices to prove the following.

Claim 10.1.2. *Suppose $Z\subseteq\omega_1$ is uncountable. Then $C_Z\subseteq f``[Z]^2$.*

Proof. Let $\delta\in C_Z$ be arbitrary. Since Z is unbounded in ω_1, let us pick some $\beta\in Z$ above δ. We now aim at finding some $\alpha\in Z\cap\delta$ such that $f(\alpha,\beta)=\delta$.

Put $n:=e_\beta(\delta)$ and $g:=r_\beta\restriction n$. Then $g\subseteq r_\beta$, so that $\beta\in B^Z_g$. In particular, $B^Z_g\nsubseteq\delta<\beta$. So, by $\delta\in C_Z$, we have $|B^Z_g|=\aleph_1$.

For each $\gamma\in B^Z_g\setminus\{\beta\}$, put $m_\gamma:=\Delta(\gamma,\beta)$. Clearly, $m_\gamma\geq n$. Evidently, there exists some uncountable subset $B\subseteq B^Z_g\setminus\{\beta\}$ such that m_γ equals to some fixed $m<\omega$ for all $\gamma\in B$.

By shrinking further, we may also assume that $r_\gamma\restriction m+1=h$ for some fixed $h:m+1\rightarrow 2$ for all $\gamma\in B$. In particular, $h\restriction m=r_\beta\restriction m$ and $h(m)\neq r_\beta(m)$.

By $B^Z_h \supseteq B$, we have $|B^Z_h| = \aleph_1$, and so by $\delta \in C_Z$, we get that $\sup(B^Z_h \cap \delta) = \delta$. In particular, δ is a limit ordinal. Let $F := \{\gamma < \delta \mid e_\beta(\gamma) \leq m\}$. Since e_β is an injection, F is finite and $\sup(F) < \delta$, and so there exists some $\alpha \in B^Z_h \cap \delta$ with $F \subseteq \alpha$. We claim that α works.

Indeed, by $\alpha \in B^Z_h$, we have $r_\alpha \restriction (m+1) = h$, so that $\Delta(\alpha, \beta) = m \geq n = e_\beta(\delta)$. In particular, $\delta \in \Gamma(\alpha, \beta)$. We need to show that δ is minimal in that sense. Let $\gamma \in \Gamma(\alpha, \beta)$ be arbitrary. Then $e_\beta(\gamma) \leq m$. If, in addition, $\gamma < \delta$, then $\gamma \in F$, but then $\gamma < \alpha$, contradicting the fact that $\Gamma(\alpha, \beta) \cap \alpha = \emptyset$. □

This completes the proof. □

A finer analysis of the preceding function establishes the following.

Exercise 10.2. *There exists a coloring $c : [\omega_1]^2 \rightarrow \omega_1$ satisfying the following strong form of* $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$.

For every $m < \omega$, every uncountable family $\mathcal{A} \subseteq [\omega_1]^m$ of pairwise disjoint sets, and every color $\alpha < \omega_1$, there exist an infinite $\mathcal{A}' \subseteq \mathcal{A}$ and $b \in \mathcal{A}$ such that for all $a \in \mathcal{A}'$:

- $\max(a) < \min(b)$, *and*
- $c(a(i), b(i)) = \alpha$ *whenever $i < m$.*[1]

Next, we shall prove the following well-known lemma.

Lemma 10.3. *There exists a function $h : [\omega_1]^2 \rightarrow \omega$ such that:*

(1) *h is locally small, that is, $\{\alpha < \beta \mid h(\alpha, \beta) \leq k\}$ is finite for all $k < \omega$ and $\beta < \omega_1$;*
(2) *h is subadditive of the first kind, that is, $h(\alpha, \gamma) \leq \max\{h(\alpha, \beta), h(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.*

Proof. For every $\gamma < \omega_1$, fix a surjection $\varphi_\gamma : \omega \rightarrow \gamma$. We first define a matrix of finite sets $\langle A^i_\gamma \mid \gamma < \omega_1, i < \omega \rangle$ by recursion on γ. Put $A^i_0 := \emptyset$ for all $i < \omega$. Now, suppose that $\gamma < \omega_1$ is an ordinal such that for all $\delta < \gamma$, $\{A^i_\delta \mid i < \omega\}$ is a chain of finite sets converging to δ. For all $i < \omega$, put:

$$A^i_\gamma := \varphi_\gamma[i] \cup \bigcup\{A^i_\delta \mid \delta \in \varphi_\gamma[i]\}.$$

Then $\{A^i_\gamma \mid i < \omega\}$ is a chain of finite sets converging to γ.

Now, define $h : [\omega_1]^2 \rightarrow \omega$ by letting for all $\alpha < \beta < \omega_1$:

$$h(\alpha, \beta) := \min\{i < \omega \mid \alpha \in A^i_\beta\}.$$

As $\{\alpha < \beta \mid h(\alpha, \beta) \leq k\} = A^k_\beta$ for all $k < \omega$, the function h is locally small.

[1] Recall that $a(i)$ stands for the unique $\alpha \in a$ with $\mathrm{otp}(a \cap \alpha) = i$.

We now prove that $h(\alpha,\gamma) \le \max\{h(\alpha,\beta), h(\beta,\gamma)\}$ holds for all $\alpha < \beta < \gamma < \omega_1$, by induction on $\gamma < \omega_1$.

Suppose that $\gamma < \omega_1$ and that subadditivity (of the first kind) holds for all $\gamma' < \gamma$. Let $\alpha < \beta < \gamma$ be arbitrary. Write $i := \max\{h(\alpha,\beta), h(\beta,\gamma)\}$. By $\beta \in A^i_\gamma$, there are two cases to consider:

► If $\beta \in \varphi_\gamma[i]$, then by definition of A^i_γ, we have $A^i_\beta \subseteq A^i_\gamma$. So, by $\alpha \in A^i_\beta$, we have $\alpha \in A^i_\gamma$, and hence $h(\alpha,\gamma) \le i$.

► If $\beta \in A^i_\delta$ for some $\delta \in \varphi_\gamma[i]$, then $h(\beta,\delta) \le i$, and by the inductive hypothesis, $h(\alpha,\delta) \le \max\{h(\alpha,\beta), h(\beta,\delta)\} \le i$. So, $\alpha \in A^i_\delta$, and hence $\alpha \in A^i_\gamma$. That is, $h(\alpha,\gamma) \le i$. □

The next theorem states that there exists a function that — in some setup — can decompose a concatenated sequence back into its original components.

Theorem 10.4 (Rinot, 2014)**.** *There exists a coloring $d : {}^{<\omega}\omega_1 \to \omega$, such that for every sequence $\langle (u_\alpha, v_\alpha, \rho_\alpha) \mid \alpha \in \Gamma\rangle$, with*

(1) *Γ is an unbounded subset of ω_1;*
(2) *u_α and v_α are nonempty finite subsets of ${}^{<\omega}\omega_1$;*
(3) *$\alpha \in \mathrm{Im}(\eta)$ for all $\eta \in u_\alpha$;*
(4) *$\rho_\alpha{}^\frown\langle\alpha\rangle \sqsubseteq \rho$ for all $\rho \in v_\alpha$,*

there exist $\alpha < \beta$ both from Γ satisfying $d(\eta{}^\frown\rho) = \ell(\eta)$ for all $\eta \in u_\alpha$ and $\rho \in v_\beta$.

Proof. Let $\langle f_\alpha : \omega \to 2 \mid \alpha < \omega_1\rangle$ be a sequence of pairwise distinct functions. Write $\Delta(\alpha,\alpha) := \infty$, and $\Delta(\alpha,\beta) := \min\{i < \omega \mid f_\alpha(i) \ne f_\beta(i)\}$ for distinct $\alpha,\beta < \omega_1$. Let $h : [\omega_1]^2 \to \omega$ be given by Lemma 10.3. For every nonconstant sequence $\sigma \in {}^{<\omega}\omega_1$, let

- $\mathcal{D}_\sigma := \{(i,j) \mid i < j < \ell(\sigma)\ \&\ \forall j^* < j(\sigma(j^*) < \sigma(j))\}$;
- $\Delta_\sigma := \max\{\Delta(\sigma(i),\sigma(j)) \mid i < j < \ell(\sigma)\ \&\ \sigma(i) \ne \sigma(j)\}$;
- $\mathfrak{m}_\sigma := \max\{h(\sigma(i),\sigma(j)) \mid (i,j) \in \mathcal{D}_\sigma\}$, provided that $\mathcal{D}_\sigma \ne \emptyset$.
- $\mathcal{P}_\sigma := \{(i,j) \in \mathcal{D}_\sigma \mid h(\sigma(i),\sigma(j)) = \mathfrak{m}_\sigma\}$, provided that $\mathcal{D}_\sigma \ne \emptyset$.
- $j_\sigma := \min\{j \mid \exists i\ (i,j) \in \mathcal{P}_\sigma\}$, provided that $\mathcal{D}_\sigma \ne \emptyset$. Otherwise, let $j_\sigma := 0$.

Let $c : [\omega_1]^2 \to \omega_1$ be given by Exercise 10.2. Given a nonconstant sequence $\sigma \in {}^{<\omega}\omega_1$, we find the least (say, in lexicographic order) pair (i,j) such that $\Delta(\sigma(i),\sigma(j)) = \Delta_\sigma$, and let $n_\sigma := c(\sigma(i),\sigma(j)))$. Finally, put

$$d(\sigma) := \begin{cases} j_\sigma - n_\sigma, & \text{if } j_\sigma > n_\sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Now, suppose that $\langle (u_\alpha, v_\alpha, \rho_\alpha) \mid \alpha \in \Gamma\rangle$ is as in the statement of the theorem. For every $\delta \in \Gamma$, write $a_\delta := \bigcup\{\mathrm{Im}(\sigma) \mid \sigma \in u_\delta \cup v_\delta\}$. By an iterative application of the pigeonhole principle and the Δ-system lemma, we may assume the existence of m, m', k, t, z, n^* such that for all $\delta \in \Gamma$:

(a) $\mathrm{otp}(a_\delta) = m$, and $a_\delta(m') = \delta$;
(b) $\max(\Delta``[a_\delta]^2 \cup h``[a_\delta]^2) = k$;
(c) $\langle f_{a_\delta(i)} \restriction (k+1) \mid i < m\rangle = t$;
(d) $\{a_\delta \mid \delta \in \Gamma\}$ forms a Δ-system with root z; $\sup(z) < \min(a_\alpha \setminus z) \leq \sup(a_\alpha) < \min(a_\delta \setminus z)$ for all $\alpha < \delta$ in Γ;
(e) $\min\{n \mid (\rho_\delta{}^\frown\langle\delta\rangle)(n) > \sup(z)\} = n^*$.

Claim 10.4.1. *There exist $\alpha < \beta$ both in Γ such that for all $\eta \in u_\alpha$ and $\rho \in v_\beta$:*

(1) $\mathcal{D}_{\eta^\frown\rho} \neq \emptyset$;
(2) $\mathfrak{m}_{\eta^\frown\rho} > k$;
(3) $n_{\eta^\frown\rho} = n^*$.

Proof. Put $\mathcal{A} := \{a_\delta \setminus z \mid \delta \in \Gamma \setminus z\}$. By the choice of c, we now find an infinite $\mathcal{A}' \subseteq \mathcal{A}$ and $b \in \mathcal{A}$ such that $\max(a) < \min(b)$ and $c(a(i), b(i)) = n^*$ for all $a \in \mathcal{A}'$ and $i < \mathrm{otp}(a)$. Since $\{\alpha < \beta \mid h(\alpha) \leq k\}$ is finite for all $\beta \in b$, and since b is finite, let us pick $a \in \mathcal{A}'$ such that $\min(h[a \times b]) > k$.

Pick $\alpha, \beta \in \Gamma$ such that $a = a_\alpha \setminus z$ and $b = a_\beta \setminus z$. By Clause (d) above, we have $\alpha < \beta$ and $a_\alpha \cap a_\beta = z$. Then, $c(a_\alpha(i), a_\beta(i)) = n^*$ whenever $\mathrm{otp}(z) \leq i < m$.

Suppose that $\eta \in u_\alpha$ and $\rho \in v_\beta$ are given, and write $\sigma := \eta^\frown\rho$.

(1) Since $\alpha \in \mathrm{Im}(\eta)$ and $\beta \in \mathrm{Im}(\rho)$, we infer the existence of a pair (i', j') such that i' is the least to satisfy $\sigma(i') = \alpha$, and j' is the least to satisfy $\sigma(j') > \sup(a_\alpha)$.[2] Then $(i', j') \in \mathcal{D}_\sigma$ and $(\sigma(i'), \sigma(j')) \in (a_\alpha \setminus z) \times (a_\beta \setminus z)$.

(2) By $(i', j') \in \mathcal{D}_\sigma$ and $(\sigma(i'), \sigma(j')) \in (a_\alpha \setminus z) \times (a_\beta \setminus z)$, we have

$$\mathfrak{m}_\sigma \geq h(\sigma(i'), \sigma(j')) \geq \min(h[(a_\alpha \setminus z) \times (a_\beta \setminus z)]) = \min(h[a \times b]) > k.$$

(3) Note that since α, β are distinct elements of $\mathrm{Im}(\sigma)$, we get that $\Delta_\sigma \geq \Delta(\alpha, \beta)$. By Clause (a) above, $\alpha = a_\alpha(m')$ and $\beta = a_\beta(m')$, and so, by Clause (c) above, $\Delta(\alpha, \beta) > k$. In particular, $\Delta_\sigma > k$. So, by Clause (b) above, we should restrict our attention to the set

$$\mathcal{I} := \{(i, j) \in m \times m \mid \Delta(a_\alpha(i), a_\beta(j)) = \Delta_\sigma\}.$$

Let (i, j) denote an arbitrary pair from $\mathcal{I}$. By $k < \Delta_\sigma < \infty$, we have $k < \Delta(a_\alpha(i), a_\beta(j)) < \infty$. By Clause (c) above, $f_{a_\alpha(j)} \restriction (k+1) = t(j) = f_{a_\beta(j)} \restriction (k+1)$, and so if $i \neq j$, then $\Delta(a_\alpha(i), a_\alpha(j)) > k$,

[2] Recall Clause (d).

contradicting Clause (b) above. It follows that if $(i,j)\in\mathcal{I}$, then $i=j$. Let $(i,i)\in\mathcal{I}$ be arbitrary. By $\Delta(a_\alpha(i),a_\beta(i))=\Delta_\sigma<\infty$, we get from Clause (d) above that $i\geq\mathrm{otp}(z)$, and then the choice of α,β entails that $c(a_\alpha(i),a_\beta(i))=n^*$. Altogether, $n_\sigma=n^*$. $\square$

Let $\alpha<\beta$ be given by the preceding claim. To see that α,β are as sought, let $\eta\in u_\alpha$ and $\rho\in v_\beta$ be arbitrary. Write $\sigma:=\eta^\frown\rho$.

Claim 10.4.2. *Suppose $(i,j)\in\mathcal{P}_\sigma$. Then:*

(1) *$\sigma(i)\in a_\alpha\setminus z$, and $\sigma(j)\in a_\beta\setminus z$;*
(2) *$j\geq\ell(\eta)$ and $i<\ell(\eta)$;*
(3) *$(i,\ell(\eta)+n^*)\in\mathcal{P}_\sigma$.*

Proof. (1) By $\sigma=\eta^\frown\rho$, we have $\{\sigma(i),\sigma(j)\}\subseteq\mathrm{Im}(\eta)\cup\mathrm{Im}(\rho)\subseteq a_\alpha\cup a_\beta$.

If $\{\sigma(i),\sigma(j)\}\subseteq a_\alpha$, then $h(\sigma(i),\sigma(j))\leq\sup(h``[a_\alpha]^2)\leq k$. Likewise, if $\{\sigma(i),\sigma(j)\}\subseteq a_\beta$, then $h(\sigma(i),\sigma(j))\leq\sup(h``[a_\beta]^2)\leq k$. However, $h(\sigma(i),\sigma(j))=\mathfrak{m}_\sigma>k$. So, by $i<j$, we get that $\sigma(i)\in a_\alpha$ and $\sigma(j)\in a_\beta$. Moreover, if $\sigma(j)\in z$, then $\{\sigma(i),\sigma(j)\}\subseteq a_\alpha\cup z=a_\alpha$, contradicting the fact that $\sup(h``[a_\alpha]^2)<\mathfrak{m}_\sigma$. So, $\sigma(j)\in a_\beta\setminus z$. Likewise, $\sigma(i)\in a_\alpha\setminus z$.

(2) By $\sigma(j)\in a_\beta\setminus z=a_\beta\setminus a_\alpha\subseteq a_\beta\setminus\mathrm{Im}(\eta)$, we infer that $j\geq\ell(\eta)$. Likewise, $\sigma(i)\in a_\alpha\setminus z\subseteq a_\alpha\setminus\mathrm{Im}(\rho)$ and $i<\ell(\eta)$.

(3) Write $j^*:=\ell(\eta)+n^*$. Then, $i<\ell(\eta)\leq j^*$. By Clauses (e) and (d) above, we have $\sigma[\ell(\eta)+n^*]\subseteq a_\alpha\cup z\subseteq\sigma(j^*)$. Consequently, $(i,j^*)\in\mathcal{D}_\alpha$.

Towards a contradiction, suppose that $h(\sigma(i),\sigma(j))>h(\sigma(i),\sigma(j^*))$. In particular, $\sigma(j)\neq\sigma(j^*)$.

▶ If $\sigma(j)<\sigma(j^*)$, then by $(i,j)\in\mathcal{D}_\sigma$, it cannot be that $j^*<j$. Then, by Clause (2), we have $\ell(\eta)\leq j<j^*$. Consequently, there exists an $n<n^*$ such that $j=\ell(\eta)+n$, and $\sigma(j)=\rho_\beta(n)$. By $\sigma(j)\in a_\beta\setminus z$ and Clause (d) above, $\sigma(j)>\sup(z)$. So n contradicts the minimality of n^*.

▶ If $\sigma(j^*)<\sigma(j)$, then by $(\sigma(i),\sigma(j^*))\in a_\alpha\times(a_\beta\setminus z)$ and Clause (d) above, we have $\sigma(i)<\sigma(j^*)<\sigma(j)$. Since h is subadditive of the first kind, then,

$$h(\sigma(i),\sigma(j))\leq\max\{h(\sigma(i),\sigma(j^*)),h(\sigma(j^*),\sigma(j))\}.$$

As we assume that $h(\sigma(i),\sigma(j))>h(\sigma(i),\sigma(j^*))$, we must conclude that $h(\sigma(i),\sigma(j))\leq h(\sigma(j^*),\sigma(j))\leq\sup(h``[a_\beta]^2)\leq k<\mathfrak{m}_\sigma$, contradicting the fact that $(i,j)\in\mathcal{P}_\sigma$. $\square$

By Claim 10.4.2(1), $\sigma(j_\sigma)>\sup(z)$. By Claim 10.4.2(2), $j_\sigma\geq\ell(\eta)$. Then, by minimality of n^* and Claim 10.4.2(3), we have $j_\sigma=\ell(\eta)+n^*$. So, by Claim 10.4.1(3), $d(\sigma)=j_\sigma-n_\sigma=(\ell(\eta)+n^*)-n^*=\ell(\eta)$. $\square$